Influence of frictional anisotropy on contacting surfaces during loading/unloading cycles

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Abstract

This paper presents numerical investigations on the loading and unloading of a three-dimensional body in frictional contact with a rigid foundation. The evolution of the sliding process during loading/unloading cycles is analyzed. The important case of anisotropy is examined along with the effect of the sliding rule. The solution algorithm is based on a variational inequality which combine the contact problem and the frictional problem. The numerical results of the punch problem show the hysteretic and irreversible behavior occurring when friction is anisotropic.

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Keywords: Elliptic friction criterion; Non-associated slip rule; Variational formulation; Uzawa algorithm; Cyclic loading

1. Introduction

In many tribological applications contacting surfaces are loaded not just once but for several thousands of cycles. Therefore, understanding the behavior of a contacting surface that undergoes a large number of load cycles is essential in order to get a better insight into the wearing process. Because engineering surfaces are rough, friction will take place. Further friction is known to be very sensitive to surface roughness pattern which can change from point to point. The roughness of an engineering surface is caused by the presence of protuberances called asperities and hollows.

Often friction is assumed to be constant and modelled using the isotropic Coulomb model where the friction properties are independent of the sliding direction. However, the size and the distribution of the asperities and hollows are not in practice identical everywhere through the surface. The anisotropic friction is one whose properties varies with direction of sliding. From a physical point of view, anisotropy of friction and wear results from the roughness anisotropy of contacting surfaces and from anisotropy and heterogeneity present in many materials due to their particular structure. An anisotropic surface roughness is then a surface in which highs and hollows in the surface are clearly oriented. The source of the roughness anisotropy is technological; the industrial process used to fabricate the bodies can create striations along preferential directions. In fact, most machining, finishing and superfinishing operations are directional, and machined surfaces have particular striation patterns unique to type of machining. Also specific techniques of manufacture produce a surface with anisotropic frictional properties. For a large number of machining processes, the striation directions are mutually orthogonal. For such surfaces, an orthotropic friction condition will provide a better description of the frictional behavior.

A friction model is completely defined by the friction condition which specify a set of admissible contact forces and the sliding rule which stipulates what directions of sliding are allowed. More sophisticated models which take into account the sliding path have been developed by Zmitrowicz [1]. The limit

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surface is usually assumed to be isotropic predicting a frictional behavior independent of the sliding direction. For many industrial applications, this assumption seems to be unrealistic and many experimental studies show that the frictional behavior can change drastically with the sliding direction, requiring an anisotropic model.

In papers devoted to orthotropic frictional contact problems [2,3], an associated sliding rule in the contact plane is assumed. However, there is no particular reason for preventing the sliding rule being non-associated. Furthermore, experimental evidence shows that the sliding rule can deviate significantly from the normal to the friction condition in the plane \( r_n = \text{const} \), where \( r_n \) is the contact pressure. In the sequel, the term “contact pressure” means the normal component of the contact force distribution which has a tangential component when friction exists on the contact surface. The occurrence of non-associated sliding rules is also supported by theoretical investigations carried out by Michałowski and Mróz [4]. They considered a model of rigid anisotropic asperities, and proved that, in general, a non-associated sliding rule occurs within the contact plane with a possible concavity of the limit friction surface. In their study, they suggested that a large class of orthotropic frictional behaviors can be modelled by considering elliptic limit surfaces. To represent non-associated sliding rule, an elliptic sliding potential is adopted but with a different semi-axes ratio.

The aim of this paper is to present a numerical investigation on the effects of anisotropy and the kinematics of sliding (slip rule) on the shear and normal stress distribution as they are responsible for increasing/decreasing the rate of wear and surface degradation at some location. The study is based on a cylinder in frictional contact with a planar surface subjected to the cyclic load. It can be observed that frictional properties have a significant influence on the stick zone location. The calculations are performed using the finite element code FER developed by Feng (see [5]). The software solves frictional contact problems using an algorithm used based on a variational formulation of the frictional contact law. In the next section, contact variables are defined and the unilateral contact law is presented. For bodies in contact, this law can be written in a rate form where the kinematical variable is the velocity. In Section 4, the rate form of the Signorini conditions are coupled with the sliding rule to give the complete frictional contact law for bodies in contact. A formulation based on a variational inequality is presented at the end of the section. Section 5 deals with the governing discrete relations of the considered frictional contact model. In Section 6, the discretized frictional contact problem is outlined and the F.E. discretization is presented in Section 7. The solution algorithm is discussed in Section 8. The example shown in Section 9 highlights the influence of the frictional model on the slip distribution and the stick zone.

2. Strong form of the governing equations

The cylinder, initially in contact with the flat rough surface, is loaded and unloaded according to a specific loading pro-

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   Fig. 1. Cylinder on a frictional planar surface.

gram. The basics of continuum contact mechanics are briefly addressed in this section. The details are omitted as they are not necessary for the current study (see [6]).

2.1. Kinematics and statics

Consider a deformable cylinder \( \mathcal{B} \) undergoing small displacements and small strains (Fig. 1). The body \( \mathcal{B} \) occupies the open, simply connected, bounded domain \( \Omega \subset \mathbb{R}^3 \). A material particle \( \mathcal{B} \) in \( \Omega \) (the closure of \( \Omega \)) is identified by its position vector \( \mathbf{X} \). The boundary \( \Gamma \) of \( \mathcal{B} \) is assumed to be sufficiently smooth everywhere such that an outward unit normal vector, denoted by \( \mathbf{n} \), can be defined everywhere on \( \Gamma \). The body under consideration is a cylinder in frictional contact with a planar surface \( \mathcal{P} \). At any time instant \( t \) of the loading process, represented by the interval \( \mathbf{I} = [0, T] \), the boundary \( \Gamma \) of \( \mathcal{B} \) can be divided into three disjoint parts: \( \Gamma_u \) with prescribed displacements, \( \Gamma_t \) with prescribed boundary loads, and the potential contact surface \( \Gamma_c \) where the cylinder \( \mathcal{B} \) is initially in contact with the half-plane but separation could possibly occur anywhere on \( \Gamma_c \) at some time \( t \):

\[
\Gamma = \Gamma_u \cup \Gamma_t \cup \Gamma_c.
\]

The body \( \mathcal{B} \) satisfy, at each time, the local equilibrium equations:

\[
\text{div} \, \mathbf{\sigma} + \mathbf{\tau} = \mathbf{0} \quad \text{in} \ \Omega,
\]

\[
\mathbf{n} \, \mathbf{\sigma} = \mathbf{\tau} \quad \text{on} \ \Gamma_t,
\]

where \( \mathbf{\sigma} \) is the Cauchy stress tensor and \( \mathbf{\tau} \) are prescribed body forces. The successive deformed configurations of \( \mathcal{B} \) are defined by the displacement field \( \mathbf{u}(\mathbf{X}) \) defined on \( \overline{\Omega} \), which satisfy the compatibility requirements:

\[
\mathbf{\varepsilon} = \frac{1}{2}(\nabla \mathbf{u} + (\nabla \mathbf{u})^T) \quad \text{in} \ \Omega,
\]

\[
\mathbf{u} = \mathbf{\varepsilon} \quad \text{on} \ \Gamma_u,
\]

where a superimposed dash denotes a prescribed quantity. Displacements are imposed on the upper face of the cube and the lower face corresponds to the contact surface \( \Gamma_c \). The remaining faces are stress free. The stress–strain relationship derives
The contact force distribution

\[ \sigma = \nabla V(\varepsilon). \]  

(5)

The potential \( V(\varepsilon) \) and its complementary \( W(\sigma) \) satisfy the following inequality:

\[ V(\varepsilon) + W(\sigma) \geq \varepsilon \cdot \sigma \]  

(6)

which becomes a strict equality for pairs \((\sigma, \varepsilon)\) related by the constitutive law.

### 2.2. Contact mechanics

Prior to loading, the lower face of the cylinder is in full contact with the rigid plane \( \mathcal{P} \). The loading starts with a uniform vertical motion of the upper face which will maintain the cylinder in contact with the planar surface during the subsequent loading cycles. At each time instant, contact conditions must be fulfilled and material points in contact with \( \mathcal{P} \) must not cross the half-plane. On the surface \( \mathcal{P} \), an orthogonal vector \( \mathbf{n} \) directed towards \( \mathcal{B} \) is defined (see Fig. 2). The unit normal \( \mathbf{n} \) coincide here with the unit vector associated with the \( z \)-axis of the coordinate system. Within the half-plane \( \mathcal{P} \) two unit vectors \( \mathbf{t}_x \) and \( \mathbf{t}_y \) are defined such that with \( \mathbf{n} \), they form an orthogonal local basis. Several options are possible for setting up the base vectors, depending on a particular choice of the unit tangent vectors. A natural choice for the unit tangent vectors is the principal orthotropy directions. The relative slip velocity corresponding to the cylinder velocity \( \dot{\mathbf{u}} \) since the planar surface is motionless. As a result of the external loading a contact force distribution equilibrating the loads is developed. The surface traction vector \( \mathbf{r}^s \) at a material point \( \mathcal{M} \in \Gamma_c \) is given by

\[ \mathbf{r}^s = \sigma \mathbf{n}. \]  

(7)

The contact force distribution \( \mathbf{r}^s \) acts on the half-plane \( \mathcal{P} \). According to the principle of action and reaction, the cylinder \( \mathcal{B} \) is subjected to

\[ \mathbf{r} = -\sigma \mathbf{n}. \]  

(8)

In the local coordinate system defined by the plane \( \mathcal{P} \) and the normal \( \mathbf{n} \), any variable \( \dot{\mathbf{u}} \) or \( \mathbf{r} \) may be uniquely decomposed into normal and tangential components according to

\[ \dot{\mathbf{u}} = \dot{\mathbf{u}}_n + \dot{\mathbf{u}}_t \mathbf{n}, \quad \dot{\mathbf{u}}_n, \dot{\mathbf{u}}_t \in \mathcal{P}, \quad \dot{\mathbf{u}}_n \in \mathbb{R}, \]  

(9)

\[ \mathbf{r} = \mathbf{r}_n + \mathbf{r}_t \mathbf{n}, \quad \mathbf{r}_n, \mathbf{r}_t \in \mathcal{P}, \quad r_n \in \mathbb{R}. \]  

(10)

The frictional force \( \mathbf{r}_f \) is dissipative and always opposes sliding (see Fig. 2). To emphasis that the dissipation is positive, the velocity \( \dot{\mathbf{u}} \) will be preceded by a “minus” sign (\( -\dot{\mathbf{u}} \)). The unilateral contact condition requires that material points \( \mathbf{X} \in \Gamma_c \) are either in contact or not in contact with the plane \( \mathcal{P} \). Therefore, the body \( \mathcal{B} \) is allowed to separate but not to cross the plane \( \mathcal{P} \). This condition constrains the displacement of the body \( \mathcal{B} \) by requiring that the normal component of the displacement of material points located at the boundary \( \Gamma_c \) is non-negative. In the problem under consideration, the lower face of the cube remains in contact with the planar surface during the whole loading process. Using the above decomposition and taken into account that the initial gap (distance between the cylinder and the planar surface projected on the normal) is null, the non-penetration condition can be expressed by

\[ -u_n \leq 0. \]  

(11)

A dual relation involves the contact pressure \( r_n \) acting on the block which must be positive \( (r_n \geq 0) \), where there is contact and zero where there is no contact. This condition is often referred to the non-adhesion condition. This set of relations may be summarized by the so-called Signorini conditions:

\[ -u_n \leq 0, \quad r_n \geq 0, \quad u_n r_n = 0 \]  

(12)

which has to be satisfied at each time instant \( t \in \mathcal{I} \). In the more general case of two bodies that may come into contact, the previous relations must take into account an initial normal gap existing between contacting bodies. This normal gap is determined by calculating the proximal points (nearest point). In the case of persistent contact \( (u_n = 0) \), the unilateral contact law can be formulated in a rate form:

\[ -\dot{u}_n \leq 0, \quad r_n \geq 0, \quad \dot{u}_n r_n = 0 \text{ on } \Gamma_c. \]  

(13)

The relation (13a) imposes that bodies in contact must either remain in contact \( (\dot{u}_n = 0) \) or must separate \( (\dot{u}_n > 0) \). The rate formulation of the Signorini conditions (13) can be combined with the sliding rule to derive the full frictional contact law applicable to material points of \( \Gamma_c \) already in contact. This complete law specifies allowable velocities of these points such that impenetrability, non-adhesion and the sliding rule are satisfied. In a more general situation, a positive gap may appear \( (u_n > 0) \). In that case, the normal relative velocity is arbitrary \( (\dot{u}_n \in \mathbb{R}) \) and the normal reaction force is equal to zero \( (r_n = 0) \). The equations developed above need to be completed by a set of relations specifying the friction model and the slip rule.

### 3. Elliptical friction models

A rate-independent friction model with a linear dependence of the limit tangential force on the normal force is considered. A theoretical investigation on friction surfaces and sliding rules
The generic form of such convex friction criterion is given by

\[ f(r_x, r_y, r_n) = \| r_t \|_\mu - r_n = 0, \tag{14} \]

where \( \| \cdot \|_\mu \) denotes the elliptic norm

\[ \| r_t \|_\mu = \sqrt{\left(\frac{r_x}{\mu_x}\right)^2 + \left(\frac{r_y}{\mu_y}\right)^2}. \tag{15} \]

The coefficients \( \mu_x \) and \( \mu_y \) are the principal friction coefficients. Curve (13) intersects the \( x \)-axis at \( \mu_x r_n \) and \( -\mu_y r_n \); it intersects the \( y \)-axis at \( \mu_y r_n \) and \( -\mu_y r_n \) (Fig. 3). Introducing the friction coefficients matrix \( M \), defined by

\[ M = \begin{bmatrix} \mu_x & 0 \\ 0 & \mu_y \end{bmatrix}, \]

the elliptic norm (15) used in the friction cone definition (14) can be replaced by the usual Euclidean norm \( \| \cdot \| \) of the transformed friction forces \( M^{-1} r_t \):

\[ \| r_t \|_\mu = \| M^{-1} r_t \|. \]

The classical isotropic Coulomb’s friction criterion is recovered by setting \( \mu_x = \mu_y = \mu \) so \( M \) is the unit matrix. The set \( K_\mu \) of allowable contact forces \( r \), defined by

\[ K_\mu = \{ r \in \mathbb{R}^3 \| r_n \| - r_n \leq 0 \}, \tag{16} \]

is convex. Its boundary and interior are denoted “bd \( K_\mu \)” and “int \( K_\mu^\circ \)”, respectively. Now it is appropriate to introduce the cone \( K_\mu^\circ \) dual (or polar) to \( K_\mu \). This set will be used later in the paper. By definition the polar cone \( K_\mu^\circ \) is the set comprising all vectors \( v \in \mathbb{R}^3 \) satisfying the following inequality:

\[ v \in \mathbb{R}^3, \quad r \cdot v \leq 0 \quad \forall r \in K_\mu. \tag{17} \]

where the dot “\( \cdot \)” stands for the usual scalar product. The scalar product in (17) is developed in the following manner

\[ r \cdot v = r_x v_x + r_y v_y + r_n v_n \geq \| r_t \|_\mu \| v_t \|_\mu^* + r_n v_n \]

\[ \geq r_n (\| v_t \|_\mu^* + v_n), \tag{18} \]

where the Cauchy–Schwartz inequality has been used to obtain the first inequality and the friction criterion (14) has been used to obtain the second one. The normal reaction \( r_n \) being positive, the inequality (18) is satisfied if

\[ \| v_t \|_\mu^* + v_n \leq 0, \tag{19} \]

where the norm \( \| \cdot \|_\mu^* \) dual of \( \| \cdot \|_\mu \), is given by

\[ \| v_t \|_\mu^* = \sqrt{(\mu_x v_x)^2 + (\mu_y v_y)^2} = \| M v_t \|. \tag{20} \]

All vectors \( v \) satisfying (19) belongs to \( K_\mu^\circ \):

\[ K_\mu^\circ = \{ v \in \mathbb{R}^3 \| v_t \|_\mu^* + v_n \leq 0 \}. \tag{21} \]

3.2. Non-associated sliding rule

The convex slip potential has also elliptical level curves but with different axis ratio (Fig. 3) and is defined by

\[ g(r_x, r_y) = \| r_t \| - \zeta \leq 0 \tag{22} \]

in which \( \| r_t \|_p \) is given by

\[ \| r_t \|_p = \sqrt{\left(\frac{r_x}{p_x}\right)^2 + \left(\frac{r_y}{p_y}\right)^2} = \| P^{-1} r_t \|. \tag{23} \]
with
\[ P = \begin{bmatrix} p_x & 0 \\ 0 & p_y \end{bmatrix} \] (24)
and \( \zeta \) is a constant whose magnitude is irrelevant. The semi-axes ratio of the slip potential is related to that of the friction condition by the following relation
\[ \frac{p_y}{p_x} = (\frac{\mu_y}{\mu_x})^k. \] (25)

In the general case, we have \( k \neq 1 \), which leads to a non-associated sliding rule
\[ -\dot{u}_n = 0, \] (26)
\[ -\dot{u}_{t_x} = \frac{\zeta}{\xi} \frac{\partial \bar{g}}{\partial r_{t_x}} = \frac{\zeta}{\xi} r_{t_x} \frac{p_x}{p_y} \| r_t \|_p, \] (27)
\[ -\dot{u}_{t_y} = \frac{\zeta}{\xi} \frac{\partial \bar{g}}{\partial r_{t_y}} = \frac{\zeta}{\xi} r_{t_y} \frac{p_y}{p_x} \| r_t \|_p, \] (28)
where the multiplier \( \dot{\lambda} \) is equal to
\[ \dot{\lambda} = \sqrt{p_x^2 \dot{u}_{t_x}^2 + p_y^2 \dot{u}_{t_y}^2} = \| \dot{u}_t \|_p^p. \] (29)

Equivalently, the sliding rule can be written as
\[ -\dot{u}_{t_x} = \frac{\dot{\lambda}}{\xi} r_{t_x} \frac{p_x}{p_y} \| r_t \|_p, \] (30)
\[ -\dot{u}_{t_y} = \frac{\dot{\lambda}}{\xi} r_{t_y} \frac{p_y}{p_x} \| r_t \|_p. \]

In this case, the multiplier \( \dot{\lambda}' \) is equal to
\[ \dot{\lambda}' = \| Q(\dot{u}_t) \|_p^p, \] (31)
where
\[ Q = P M^{-1} = M^{-1} P = \begin{bmatrix} p_x & 0 \\ 0 & p_y \end{bmatrix}. \] (32)

is called the sliding non-associativity matrix.

4. Frictional contact law

The frictional contact law aims at describing contact interactions. To derive this relationship, we add the geometric constraints (impenetrability condition) to the friction law. The sliding rule is then combined with the rate form of the unilateral contact conditions so we obtain the frictional contact law.

4.1. Analytical formulation

The complete form of the frictional contact law deals with the three possible physical situations, which are separation, contact with sticking, and contact with sliding. Mechanical dissipation occurs only for the last case. This law is applicable only to material points in contact. Two overlapped “if...then...else” statements can be used to write it analytically (Box 1):

\[ \text{Box 1} \]

if \( r_n = 0 \) then
  ! sticking
  \[ -\dot{u}_n \leq 0 \]
else if \( r \in \text{int} K_\mu \) then
  ! sticking
  \[ \dot{u}_n = 0 \] and \( \dot{u}_t = 0 \)
else if \( r \in \text{bd} K_\mu \) and \( r_n > 0 \)
  ! sliding
  \[ \dot{u}_n = 0 \] and \( -\dot{u}_t = \frac{\zeta}{\xi} \frac{r_{t_t}}{p_y} \| r_t \|_p \), \( \zeta > 0 \)
endif

In the first and the second part of the statement, the multi-valued character of the frictional contact constitutive model is revealed. If \( r_n \) is null then the velocity \( \dot{u}_t \) is arbitrary but its normal component \( \dot{u}_n \) should be positive. In other words, an infinite number of velocity vectors \( \dot{u}_t \) are related to the single contact force \( r = 0 \). Further, if \( -\dot{u}_t \) is null then the reaction \( r \) should be in \( K_\mu \) but its direction or magnitude are not specified. They are arbitrary. Again, one element of \( \mathbb{R}^3 \) (\( -\dot{u}_t = 0 \)) can be related to an infinite number of \( r \in \mathbb{R}^3 \). The inverse constitutive frictional contact law, i.e. the relationship \( r(-\dot{u}_t) \) can be written as (Box 2):

\[ \text{Box 2} \]

if \( \dot{u}_n > 0 \) then
  ! separation
  \[ r_n = 0 \]
else if \( \dot{u}_t = 0 \) then
  ! sticking
  \[ r \in K_\mu \]
else if \( \dot{u}_t \in \text{T} - \{0\} \)
  ! sliding
  \[ r_n > 0 \] and \( r_{t_t} = -r_n \frac{p_y^2 \dot{u}_{t_t}}{\| Q(-\dot{u}_t) \|_p} \)
endif

4.2. Inequality-based formulation

The previous forms of the frictional contact law, given in Boxes 1 and 2, could be very well used in numerical implementations where the contact problem is dissociated from the friction problem. This class of algorithm has been very much developed. However one can expect that a coupled algorithm where both problems are solved in a single step could improve the computational performances and robustness. With this aim in mind, the frictional contact law need to be reformulated. The main idea behind this reformulation is to provide a natural...
basis for the application of the Uzawa algorithm where a single projection will be performed.

The slip rule, as written in relations (26)–(28), exhibits a structure similar to the non-associated flow rule in plasticity even if the slip rule is associated. Indeed, during sliding, contact is maintained. Therefore the normal velocity, which is equal to zero (\( \dot{u}_n = 0 \)), is not related to the normal component of the reaction \( r_n \) through normality. In fact, if we regard the contact force \( r \) and the velocity \(-\dot{u} \) as conjugate quantities of each other and consider an associate slip rule, the normality will not occur since it would require that the velocity would have a normal separating component.

During sliding, the components of the velocity vector are given by

\[
\begin{align*}
\dot{u}_n &= 0, \quad -Q^2(\dot{u}_r) = \lambda \frac{\partial f(r)}{\partial r}, \\
\end{align*}
\]

but can be rewritten in the following obvious way:

\[
\begin{align*}
-(\dot{u}_n + \dot{\lambda}) &= \lambda \frac{\partial f}{\partial r}, \\
-Q^2(\dot{u}_r) &= \frac{\partial f}{\partial r} = \frac{\partial f}{\partial r} \cdot \frac{\partial f}{\partial r},
\end{align*}
\]

where \( \lambda \) is obtained after eliminating the tangential reaction components from the slip rule:

\[
\lambda = \|Q^2(\dot{u}_r)\|^{\ast}.
\]

Indeed the first relation of (34) yields back \(-\dot{u}_n = 0 \), that is during sliding contact must hold. An obvious vector addition gives:

\[
-(Q^2(\dot{u}_r) + (\dot{u}_n + \dot{\lambda}) n) = \dot{\lambda} \text{ grad } f.
\]

The gradient vector is unique everywhere except at the apex \((r = 0)\) where the friction criterion is not differentiable. The non-differentiability issue is addressed simply by replacing the gradient operator by the subgradient one (see [8]):

\[
-(Q^2(\dot{u}_r) + (\dot{u}_n + \|Q^2(\dot{u}_r)\|^{\ast} n) \in \partial f(r).
\]

The above relationship, called differential inclusion, is a condensed formulation of the frictional contact relationship which is appropriate everywhere on the friction cone, including the apex. Whenever the \( f(r) \) is differentiable, the subdifferential operator will coincide with the gradient operator. Let us introduce now the following vector:

\[
v = v_r + v_n n
\]

with

\[
v_r = -Q^2(\dot{u}_r), \quad v_n = - (\dot{u}_n + \|Q^2(\dot{u}_r)\|^{\ast})
\]

and rewrite (39) as

\[
v \in \partial f(r).
\]

The differential inclusion (40), is equivalent to the following inequality, known as the convexity inequality for non-differentiable function (see [8]):

\[
f(r') \geq f(r) + (r' - r) \cdot v \quad \forall r' \in \mathbb{R}^3.
\]

Taking into account that

\[
f(r') \leq f(r)
\]

inequality (41) reduces to

\[
r \in K_{\mu} : (r - r') \cdot v \geq 0 \quad \forall r' \in K_{\mu}.
\]

In particular, at the apex \((r = 0)\) we have

\[
r' \cdot v \leq 0 \quad \forall r' \in K_{\mu}.
\]

Accordingly at the apex, any vector \( v \) belonging the cone \( K_{\mu}^{*} \) is an admissible velocity, i.e. it is related to \( r = 0 \) through the frictional contact law. Indeed, it can be easily checked by combining the definition of \( v \) (38)–(39) and \( K_{\mu}^{*} \) (21) that such vector does not violate the non-penetration condition. The inequality formulation of the frictional contact law (43) provides the basis to derive projection-type algorithms.

5. Discrete frictional contact law

The inequality developed in the previous section is now discretized so it can be used during the local stage where the reaction are computed. The time interval \([0, T]\) is partitioned into \( N \) sub-intervals of size \( \Delta t \), not necessarily equal, according to

\[
0 = t_0 < t_1 < \cdots < t_{n-1} < t_n < \cdots < t_N = T.
\]

We set \( t_n = \omega(t_n) \) and \( \Delta \omega = \omega_n - \omega_{n-1} \), where \( \omega \) represents any variable. Between two time instants, the velocity is assumed to be constant. In order to ensure convergence and stability requirements, the implicit scheme is considered. As a result of a backward-Euler-type approximation of (43), the frictional contact law is satisfied at the end of each time step:

\[
\begin{align*}
\tau^{n+1} &\in K_{\mu} \quad \text{ such that } \\
\Delta v \cdot (r' - r^{n+1}) &\leq 0 \quad \forall r' \in K_{\mu},
\end{align*}
\]

where the components of the vector \( \Delta v \) are given by

\[
\Delta v_r = -Q^2 \Delta u_r, \quad \Delta v_n = -(\Delta u_n + \|Q^2 \Delta u_r\|^{\ast}).
\]

The previous inequality can be transformed into a projection inequality

\[
\begin{align*}
\text{Find } r^{n+1} &\in K_{\mu} \quad \text{ such that } \\
(r^{n+1} - \tau) \cdot (r' - r^{n+1}) &\geq 0 \quad \forall r' \in K_{\mu},
\end{align*}
\]

where the vector \( \tau \) is given by

\[
\tau = \begin{bmatrix} \tau_r \\ \tau_n \end{bmatrix}
\]

with

\[
\tau_r = r^{n+1} - \rho Q^2 \Delta u_r
\]

and

\[
\tau_n = r^{n+1} - \rho (\Delta u_n + \|Q^2 \Delta u_r\|^{\ast}).
\]
The last inequality (46) means that the reaction at the end of the time step is the projection of the augmented surface traction \( \tau \) onto the convex Coulomb's cone \( K_\mu \):

\[
r^{n+1} = \text{proj}(\tau, K_\mu).
\]

(50)

Three different situations emerge according to the position of the prediction in the forces space. The first case corresponds to a prediction located in the cone \( K_\mu \). Its projection is the prediction itself, i.e. \( r^{n+1} = \tau \). The second one relates to a prediction situated in the cone \( K_\mu^* \), where its projection turns out to be the origin of the forces space, i.e. \( r^{n+1} = 0 \). In the last case, the prediction is neither in \( K_\mu \) nor in \( K_\mu^* \), and the corrector step requires computing the projection of the prediction. The projection of a point onto a convex set is equivalent to the projection of a point onto a convex set.

6. Finite-step boundary value problem and variational formulation

The solution of the elastic frictional-contact initial boundary value problem, under a given history of external actions, requires following the evolution of the body response since the frictional contact law is intrinsically path-dependent. A numerical technique, which combines a space and time discretization, is used to solve this problem. The time discretization is based on a subdivision of the external actions history into a sequence of loading conditions at selected time instants. The solution is then achieved by solving a sequence of problems in which the load increments are applied and the variables at the end of each increment are updated. We suppose that the solution of the load increments are approximated according to

\[
\Delta u = N(X) \Delta U,
\]

(61)

where \( \Delta U \) is the unknown nodal displacement increment vector, \( N(X) \) is the matrix of polynomial shape functions and the matrix \( B(X) \) is given by

\[
B(X) = \frac{\partial N(X)}{\partial X}.
\]

The compatibility conditions (55) on \( \Gamma_u \) are enforced by substituting nodal unknowns by their corresponding values. Taking into account (61), the fully discrete form of the functional is given by

\[
\int_{\Omega} \Delta W(\Delta \varepsilon, \Delta U) \, d\Omega - \int_{\Gamma} \Delta \mathbf{f} : \Delta \mathbf{u} \, d\Gamma - \int_{\Gamma_c} \mathbf{R} : \Delta \mathbf{u} \, d\Gamma = 0.
\]

(62)

where \( \Delta \mathbf{F}^{\text{ext}} \) corresponds to the generalized nodal force increment vector

\[
\Delta \mathbf{F}^{\text{ext}} = \int_{\Omega} N^T \Delta \mathbf{f} \, d\Omega + \int_{\Gamma_i} N^T \Delta \mathbf{f} \, d\Gamma
\]

(63)

and \( \Delta \mathbf{R} \) is the equivalent contact reaction increment vector at nodes. The equilibrium equations over a time step are given by

\[
\int_{\Omega} \frac{\partial \Delta W(\Delta \varepsilon, \Delta U)}{\partial \Delta U} \, d\Omega - \Delta \mathbf{F}^{\text{ext}} - \Delta \mathbf{R} = 0.
\]

(64)

The standard approach to derive the principle of virtual work over an increment consists in taking the inner product of the local equilibrium equation (51) with a test function \( \delta \mathbf{u} \) satisfying \( \delta \mathbf{u} = 0 \) on \( \Gamma_u \) and \( \delta \mathbf{u} \geq 0 \) on \( \Gamma_c \) (virtual displacement increment):

\[
\int_{\Omega} \text{div} \, r^{n+1} : \delta \mathbf{u} \, d\Omega + \int_{\Omega} \tilde{r}^{n+1} : \delta \mathbf{u} \, d\Omega = 0.
\]

(57)

Integrating by parts the first term and taking into account the boundary conditions leads to the following variational equation:

\[
\int_{\Omega} \delta \varepsilon^{n+1} : \mathbf{e} \, d\Omega - \int_{\Omega} \tilde{\mathbf{e}}^{n+1} : \delta \mathbf{u} \, d\Omega
\]

\[
- \int_{\Gamma_i} \tilde{r}^{n+1} : \delta \mathbf{u} \, d\Gamma - \int_{\Gamma_c} \mathbf{r}^{n+1} : \delta \mathbf{u} \, d\Gamma = 0.
\]

(58)

Taking into account that the stress filed is in equilibrium at \( t_n \), the weak form can be written in terms of finite increments as follows:

\[
\delta \left\{ \int_{\Omega} \Delta W(\Delta \varepsilon) \, d\Omega - \int_{\Omega} \Delta \mathbf{f} : \Delta \mathbf{u} \, d\Omega
\]

\[
- \int_{\Gamma_i} \Delta \mathbf{f} : \Delta \mathbf{u} \, d\Gamma - \int_{\Gamma_c} \mathbf{R} : \Delta \mathbf{u} \, d\Gamma \right\} = 0.
\]

(59)

If the behavior on contact surface is frictionless, the functional (59) becomes with the classical energy functional:

\[
\delta \left\{ \int_{\Omega} \Delta W(\Delta \varepsilon) \, d\Omega - \int_{\Omega} \Delta \mathbf{f} : \Delta \mathbf{u} \, d\Omega
\]

\[
- \int_{\Gamma_i} \Delta \mathbf{f} : \Delta \mathbf{u} \, d\Gamma \right\} = 0.
\]

(60)

7. Finite element discretization

The displacement increment field and the transformation gradient are approximated according to

\[
\Delta \mathbf{u} = N(X) \Delta \mathbf{U},
\]

(61)

where \( \Delta \mathbf{U} \) is the unknown nodal displacement increment vector, \( N(X) \) is the matrix of polynomial shape functions and the matrix \( B(X) \) is given by

\[
B(X) = \frac{\partial N(X)}{\partial X}.
\]
Combining the structural equilibrium equations (64) with the incremental frictional contact constitutive relation (56), the solution of the boundary value problem over a time step is obtained by solving the following system of equations:

Find $\Delta U$ and $\Delta R$ satisfying:

$$K\Delta U - \Delta F_{\text{ext}} - \Delta R = 0,$$

$$R^{n+1} = \text{proj}(\tau(\Delta U), K_\mu) \text{ on } \Gamma_c,$$

$$\Delta U = \Delta U \text{ on } \Gamma_u.$$

8. Solution algorithm

The system of equations (65)–(67) has to be solved iteratively because the displacement increment vector, the contact surface and the reactions $\Delta R$ are unknown. The solution algorithm (see Box 3) tackle separately the equilibrium equations and the contact non-linearities. At the beginning of each load step, $\Delta R$ is set equal to zero and $\Delta U$ is computed according to (66). Having an estimate of the displacement increment, the reactions have to be computed and the penetration that has occurred by assuming $\Delta R = 0$ must be corrected. This is achieved by applying the Uzawa iterative scheme as shown in Box 3. This procedure involves only variables associated with contact nodes. In contrast with classical methods (penalty, Lagrange multiplier, see [10,11]), the present algorithm does not require an update of the global stiffness matrix during the contact iterations. Furthermore the total number of degree of freedom remain unchanged.

Box 3

- Read the problem data
- Assemble the stiffness matrix $K$
- Modify $K$ for essential boundary conditions

For Each Load Step:

- Initialisation: $\Delta U^0 = 0$ and $\Delta R^0 = 0$
- Compute the external load vector: $\Delta F_{\text{ext}}$
- Detect contact node using a gap function
- Solve: $\Delta U = K^{-1}(\Delta F_{\text{ext}} + \Delta R^0)$
- On $\Gamma_c$
  - Initialisation: $\Delta R^0 = 0$
  - Contact Loop $k : 0 \rightarrow m$
    - $\Delta U^{k+1} = \mathbf{W}\Delta R^k$
    - $\Delta R^k = \text{proj}(\tau^k + (\Delta U^{k+1}), K_\mu) - R_a$
    - Convergence? $k = k + 1$
- Solve: $\Delta U = K^{-1}(\Delta F_{\text{ext}} + \Delta R)$
- Update: $U = U + \Delta U$, $R = R + \Delta R$

Next, the prediction $\tau$ is computed and projected on the cone $K_\mu$ to give the reaction (see Box 3). The Uzawa algorithm is known for being convergent but requiring quite a few iterations. The convergence rate strongly depends on the regularization factor $\rho$. A good choice is crucial to ensure that the algorithm will convergence within a few iterations. Our experience has shown that a different factor $\rho$ for each contact node gives a better convergence. In our calculations, the factor $\rho$ is calculated using the diagonal terms of the flexibility matrix:

$$\rho = \frac{1}{\min(w_{nn}, w_{tx}, w_{ty})}.$$

This choice has proven to be satisfactory for most of the cases.

9. Numerical application

The numerical strategy detailed in the previous sections has been applied successfully to a wide range of problems involving an isotropic friction condition with associated sliding rule (see Refs. [12–14]). The examples treated have shown that the algorithm is very competitive as the augmentation phase involves only one prediction-correction step. The present application is a further test of the algorithm robustness when the friction criterion is anisotropic, the slip rule is non-associated and unloading occurs. The results show the strong influence of the frictional properties on the stick zone and therefore the contact (normal and tangential) stress distribution.

The problem under consideration is a deformable elastic cylinder in contact with a rigid surface (Fig. 1). The radius and the height of the cylinder are both equal to 10 mm. The Young modulus $E$ of the cylinder is taken equal to 210 000 MPa and the Poisson ratio is 0.3. On the surface contact, the friction condition is assumed to be anisotropic. The cylinder is subdivided into 1280 eight-node brick-like elements as shown in Fig. 1. Each element has 27 integration points. The base of the cylinder is in contact with the rigid plate whose normal vector is

<table>
<thead>
<tr>
<th>Case</th>
<th>$\mu_1$</th>
<th>$\mu_2$</th>
<th>$p_x$</th>
<th>$p_y$</th>
<th>Load cycle</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>0.20</td>
<td>0.20</td>
<td>0.20</td>
<td>0.20</td>
<td>1</td>
</tr>
<tr>
<td>B</td>
<td>0.30</td>
<td>0.25</td>
<td>0.30</td>
<td>0.15</td>
<td>3</td>
</tr>
<tr>
<td>C</td>
<td>0.30</td>
<td>0.15</td>
<td>0.05</td>
<td>0.20</td>
<td>1</td>
</tr>
</tbody>
</table>

![Fig. 4. Loading program.](image)
Fig. 5. Contour plots of slip after five load steps.

Fig. 6. Contour plots of $r_n$. 

CASE A
CASE B
CASE C
(0, 0, 1). The total number of nodes in contact with the rigid plane is 337. This number seems to be large enough to represent the complexity of the frictional behavior, specially during the unloading stage. The loading process consists of a cyclic vertical rigid motion of the cylinder upper face (displacement controlled loading). Before cycling, the cylinder is compressed by imposing a vertical rigid motion to the upper surface of the cylinder. The magnitude of this vertical displacement is 0.1 mm. This displacement is applied in one step. The subsequent part of the loading process consists of one or several cycles (see Table 1). For each cycle, five load increments are performed for the loading phase while the unloading stage is executed with 10 load increments. Fig. 4 shows the loading program. Three different sets of frictional properties are considered (see Table 1). Set C corresponds to an anisotropic model with non-associated slip rule. For each set, one or several load cycle are performed (see Table 1). The analysis is performed on a PC (Pentium III 733 MHz). The convergence criterion involves the penetration which could not exceed a tolerance equal to $10^{-8}$. Fig. 5 shows the contour plots of the slip, after five load steps (peak of the first cycle), between the lower surface of the cylinder and the rigid plate for the three sets of frictional properties. The slip correspond to the Euclidean norm of the tangential displacement. For set A, the frictional model being isotropic, the iso-values of the slips are circular. As expected, the stick zone is located around the base center and sliding tends to increase towards the periphery. As it can be seen, the anisotropy (set B) of the friction condition influences significantly the slip distribution pattern. The stick area is now an ellipse with a semi-axes ratio equal to the semi-axes ratio of the friction criterion. The slips increases gradually from the stick area and are maximum on the periphery in the y-direction since for both cases $\mu_x$ is greater than $\mu_y$. Once a non-associative sliding rule is considered, the slip distribution can change drastically according to the degree of non-associativity. Indeed, if the slip rule is strongly non-associated, the iso-values of the slip become non-convex as shown in Fig. 5.

The normal reaction (see Fig. 6) is higher in the periphery of the cylinder and decreases as we approach the center of the cylinder basis. However, the decrease is not monotonic along every radius. Indeed, the normal reaction attains its minimum in four tiny areas located at approximately the third of the radius.
from the base center. For all cases considered, contour plots of the normal component of the contact reaction have similar pattern.

Fig. 7 is a further representation of the stick zone. The dark gray area corresponds to points where $f < 0$ (part of the stick zone). The light gray color indicates the place where we have $f = 0$. The stick zone is circular if friction is isotropic and elliptic for the anisotropic cases. The slip rule does not influence strongly the stick zone shape.

The unloading stage is now examined. During the first unloading stage (each unloading stage are performed using 10 steps), we will be interested in the friction criterion just after the first, third, fourth and sixth unloading step. Fig. 8 shows the contour plots of the friction criterion for each case and at each unloading step. It can be seen that the stick zone exhibits a complex pattern. Indeed, if friction is isotropic then the stick area is located at both the base center and the periphery. The stick zone located at the periphery tends to grow and to move towards the center during unloading. Whereas in case of anisotropic friction criterion, the stick zone appears to be getting thinner at the base center and at the periphery as well. Again the nature of the slip rule seems to have a significant influence on the stick area pattern. The stick zone is split into two or more area.

In order to demonstrate the hysteretic behavior occurring when friction is anisotropic (case B), we consider the motion
of a material point $P$ (see Fig. 9). The motion of the material point $P$ during two cycles of loading/unloading is shown in Fig. 9. The difference between loading path and the unloading path appears to be evident. Further, there is a slight difference between the trajectories during the first cycle and the second one. Fig. 10 depicts the evolution of the components of the friction force during a cycle. The slip versus load step is shown in Fig. 11. At each load peak, a plateau can be observed. This plateau corresponds to a stick condition, that is the point $P$ is motionless. Before the plateau, the slip increases linearly whereas it decreases in a non-linear fashion after the plateau. Fig. 12 shows the relationship between the equivalent applied force (resulting from imposed displacement) and the slip. It can be seen that the relationship is linear during loading. The unloading stage starts with a sharp drop in the applied force followed by a non-linear decrease (Fig.12).

10. Conclusions

The complex behavior occurring during loading/unloading cycles has been examined on the punch problem. The strong influence of the anisotropy and the slip rule has been demonstrated. Further developments are being undertaken, including the analysis of contact between more than one deformable bodies and the analysis of fretting problems.

References


