Eur. J. Mech. A/Solids, **17**, n° 4, 619-636, 1998

Large displacement analysis for ideally flexible sails

O. LE MAÎTRE *, J. E. SOUZA DE CURSI ** and S. HUBERSON *

ABSTRACT. – We consider the equilibrium of a sail under a given aerodynamic field of external forces. The sail is considered to be an ideally flexible structure, having the behaviour of a network of stress unilateral strings: all the internal efforts are traction efforts. This model leads to a Non Convex Optimisation Problem and a complete theory can be established, leading to relevant results of uniqueness for the field of stresses, even if configurations of equilibrium are not unique. © Elsevier, Paris

1. Introduction

A complete modelling of sail's steady equilibrium involves a fluid/structure analysis: on the one hand, the effects of the wind must be considered; on the other hand, a mechanical model of the sail itself must be introduced. Moreover, due to the complexity and the diversity of real operating conditions, the usual models are simplified ones, corresponding to particular boat motions, laminar inflow conditions or rigid motions. The basic assumptions and simplifications of each model usually underline particular aspects of the problem. For steady case, we point out that the interaction between the sail and the external flow is to be considered by a complete model: the presence of the sail modifies the flow, whilst the latter applies aerodynamic forces on the sail and modifies its geometry. This suggests a fixed-point method for the numerical simulation: a flow field is given and a configuration of the sail is computed (structural step); a new external flow is then computed by taking into account the new configuration of the sail (aerodynamic step). This leads to a new external flow, a new geometry of the sail and so on: the sequence of structural/aerodynamic steps is repeated until some stopping condition is satisfied. In this paper, we consider the structural step of this procedure and the very flexible behaviour of the sail itself is underlined: we consider the sail as an ideally flexible structure submitted to aerodynamic forces resulting from a given flow field. So, only the resulting aerodynamic forces are computed as a part of a fluid-structure interaction problem. The effects of the changes of the geometry on the flow are not considered here, so the external loads are supposed to be well approximated by that of the initial configuration.

The very flexible behaviour of sails leads to large displacement analysis of very thin structures leading to the classical models of membranes (flexion stresses are neglected). Due to large displacements, this models are geometrically nonlinear, and since deformations in modern sails (variations of lengths and angles in the material) remain low, constitutive laws of the material can be consider as linear: tensions in the structure are linear functions of the local deformations. We emphasise on a consequence of the capability of the sail to undergo large displacements without significant deformations: there can exist displacement fields that keep constant the

* GEMH, BP540, 76058 Le Havre Cedex, France
** INSA de Rouen, BP 08, 76131 M^t S^t Aignan, France

0997-7538/98/04/© Elsevier, Paris

elastic energy in the structure (changes due to variations of the curvature are neglected). For example, there are many natural configurations of the sail. By natural, we mean a configuration that presents no deformation (strains are null everywhere): the definition of a reference (natural) configuration required by the models is arbitrary.

Such geometrically nonlinear membranes have been extensively studied (see for instance theorical works in (Antman, 1995), (Le Dret *et al.*, 1995) and numerical implementation in (Zienkiewicz, 1977), but they require important computational efforts compare to sails makers resources. Moreover, we have to keep in mind that the objective of this work is the simulation of sails' responses to unsteady perturbations and nonlinear membranes have been considered too expensive (in CPU time) to be coupled with a flow model: simplified models and/or different strategies are then necessary.

Charvet (Charvet, 1992a) presents an original scheme to estimate the steady equilibrium configuration of a sail. His analysis is based on the decomposition of the displacement from a given reference configuration to the equilibrium shape as a two steps process: first, large displacements of an inextensible sail and second, small displacements of an elastic sail. The first level leads to an intermediate configuration of equilibrium, where the sail is considered as a tight fishing net: the admissible configurations are such that the distances between neighbouring nodes are invariant. At this level, there is no variation of the internal energy of the sail: the solution is the configuration which maximises the work of external forces. In the second level, the elasticity of the material is taken into account: a small displacement analysis of the preceding configuration leads to a classical linear elasticity problem and a new configuration is determined. So, this method solves firstly a nonlinear problem with geometrical constrains of in-extensibility and, secondly, a linear problem which is expected to take into account elastic deformation. This method is referred in the following as the "two levels model". It has given satisfactory results but has shown to be inefficient when the Young's modulus of the material is small and elastic deformations lead to large displacements (specially on free borders). In this case, a large elastic displacement analysis is needed.

The originality of this method is essentially the first level which furnishes an estimated (and not an arbitrary) reference configuration before considering elastic effects. This is a great improvement compare to the previous works of (Jackson, 1985), (Jackson *et al.*, 1986) and (Fukasawa *et al.*, 1992) where small displacements analysis is performed on an arbitrary shape.

In the second part of this paper, we present an alternative approach to the nonlinear membrane models. We assume that sails can be correctly considered as structures constituted of elastic strings networks (*i.e.* fabrics). This approximation of the medium using strings is equivalent to consider only elastic deformations (strains) in a finite number of given material directions. Here material direction means that it moves with the structure. The strings network approximation is a simplified form of the nonlinear membrane model and it leads to a non-convex variational equation (Principle of Virtual Works) of Minimum of the Energy, which is functionally solved by relaxation. The functional resolution leads to relevant mechanical properties, as the uniqueness of the internal efforts and the equivalence between the Principle of Virtual Works and the Principle of the Minimum of the Energy (this is not immediate for a non-convex situation). Moreover, it leads to a numerical method, which is tested in some simple situations (Finite Element Approximation of the solution and iterative solution of the nonlinear resulting equations).

The connection between the two formulations in finite displacement is investigated: in fact, for a modulus of Young going to infinity, an asymptotic analysis of the large deformation model leads to a model where the *geodesic distances* between points of the sail are invariant. For a tight sail, this corresponds to the first level of the two levels model.

The application of the considered methods to an unsteady problem is discussed.

2. The two level model

As previously observed, sails are very flexible and can be considered as ideally flexible structures. Moreover, small loads can produce large displacements, even under infinitesimal strains. Charvet (Charvet, 1992a) has proposed that the deformation of a sail can be approached by considering large displacements of an inextensible sail and subsequent small displacements of an elastic one. This 'two levels' method leads to the following method of calculation:

Level 1 (Inextensible deformation): Let be given a configuration of reference $\vec{x_R}$ and a grid defining nodes on this configuration. We compute an intermediate configuration $\vec{x_{in}}$ such that the distances between neighbouring nodes are invariant.

Level 2 (Linear Elastic deformation): The final configuration $\vec{x}$ is obtained by the equation $\vec{x} = \vec{x_{in}} + \vec{u}$, where u is assumed to be infinitesimal: a linearised elastic formulation is considered for the determination of $\vec{u}$.

2.1. Definition of the geometry

We consider a sail in a configuration $\vec{x}$. The sail is characterised by a length L_s and a thickness h supposed to be small compared to L_s. Then the complete domain Ω of the space occupied by the sail is defined by a map which associates the curvilinear coordinates $a = (a^1, a^2)$, $a \in \Omega$ defined on the mean surface to the vector $\vec{x}(a)$:

$$\vec{x} \; : \Omega \times \left[-\frac{h}{2} : \frac{h}{2}\right] \longrightarrow R^3$$

The unitary vector normal to the surface, denoted by $\vec{n}$ is defined by:

$$\begin{cases} \vec{n} = \dfrac{\partial_1 \vec{x} \wedge \partial_2 \vec{x}}{\|\partial_1 \vec{x} \wedge \partial_2 \vec{x}\|} \\ \partial_i \vec{x} = \dfrac{\partial \vec{x}}{\partial a_i} \end{cases}$$

This definition supposes that the coordinates lines $a^i = c^{te}$ are not parallel. Figure 1 presents a schematic view of the problem.

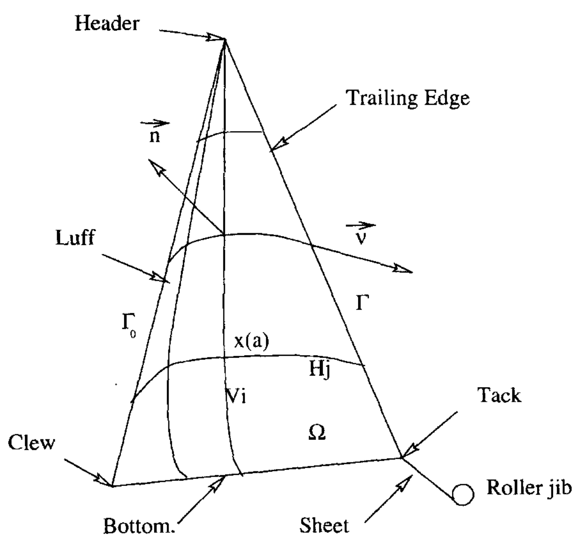

Fig. 1. – Definition of the geometry.

2.2. First level: inextensible deformation

At this level, the sail is approximated by an inextensible fishing net, *i.e.* an ideally flexible and inextensible structure such that:

1. Any configuration is defined by the position of a finite set of points (the nodes of the grid mentioned above).

2. The set of the admissible configurations is the set of the configurations which preserve the distance between the neighbouring nodes.

In this first step, Charvet assume that this approximation of the conservation of the metric on the surface, by a finite number of lengths lying on the surface furnishes an accurate estimation of constant elastic energy deformations: the variation of the internal energy of the sail is neglected: the strain tensor is identically null and only the work of the external loads (the pressure jump across both side of the surface) is taken into account. Then, the nonlinear problem of the determination of the configuration of equilibrium of the structure reduces to the problem of the maximisation of W, the work of the external forces, under geometrical constraints of in-extensibility. W is given by

$$W(\vec{X}) = \int_{sail} \vec{x}_{int}.\vec{n}\delta P ds,$$

where $\vec{n}$ is the unitary vector normal to the surface and δP the pressure jump. After discretization involving a grid having $N \times N$ nodes $z_i, i = 1, \dots, N \times N$, this functional is approached by

$$W(\vec{x}_{int}) \approx W_{ap}(\vec{x}_{int}) = \sum_{i=1}^{NN} S(z_i)\vec{x}_{int}(z_i).\vec{n}\delta P,$$

where $S(z_i) > 0$ is a given coefficient, connected to the grid: if the total area of the sail at the configuration of reference $\vec{x}_R$ is S, we have

$$\sum_{i=1}^{N\times N} S(z_i) = S\,.$$

If the neighbours of the node z_i are $z_i^1, \dots, z_i^{nne(i)}$, the in-extensibility condition reads as

$$\phi_i^j(z_i, z_i^j) = \mid \vec{x}_{int}(z_i) - \vec{x}_{int}(z_i^j) \mid - \mid \vec{x}_R(z_i) - \vec{x}_R(z_i^j) \mid = 0,$$
$$j = 1, \dots, nne(i), i = 1, \dots, N \times N\,.$$

As previously observed, this set of conditions corresponds to the discretization of the surface by a net such that neighbouring nodes constitute extremities of segments of constant length. The set of the admissible configurations is

$$C_{ad} = \{\vec{x}_{int} \mid \phi_i^j(z_i, z_i^j) = 0, j = 1, \dots, nne(i), i = 1, \dots, NN\}.$$

We emphasise that this discretisation of the geometrical constrain of conservation for the metric neglects shear stresses and then it could be improved by a discretisation of the sail using triangular elementary surfaces for which the distances between the three vertices are conserved. Unfortunately, we tested this triangular discretisation but it leads to an highly non convex problem and the nonlinear solver were not able to find any solution.

The configurations of equilibrium are the solutions of

Find the configuration $\vec{x}_{int} \in C_{ad}$ *which maximises* W_{ap} *on* C_{ad}.

This optimisation problem is solved using an iterative method involving a quadratic approximation of its Lagrangian.

We emphasise that, in this step, the internal energy of the sail is assumed to be constant: the elasticity of the material is taken into account at the second level.

2.3. Second level: linear elastic deformation

Charvet assumes that the effects of elasticity can be described by considering a linear elastic deformation of the intermediate configuration $\vec{x}_{int}$: an infinitesimal displacement formulation is introduced where $\vec{x}_{int}$, the geometry obtained in the first step, is considered as the configuration of reference. Such a linearization is classical and will not be examined in detail here.

The mechanical characteristics of the sail are given by the Young modulus (E), the Poisson coefficient (ν) and the thickness (h). The sail can be considered either as a membrane (ideally flexible), or as a thin shell (membrane and bending stresses) (see for instance (Destuynder, 1990)). The model of membrane only account for the tensions in the tangential plane while the thin shell theory also account for the variation of curvature. Within the framework of shell theory, the solution of the elastic problem involves a bi-laplacien for the displacements normal to the mean surface, and it can be computed with a finite elements method involving a third order polynomials approximation (see for instance (Ciarlet, 1978)).

The linearisation appears in the separation of the effects of tangential and normal loads. As a matter of fact, especially in our case which considers normal loads (at least for in-viscid flow assumption), numerical difficulties can occur and erroneous results can be found for nearly plane and plane surfaces (Charvet, 1992a). In such configurations, coupling the structure and the flow problems can lead to instabilities and it is then necessary to introduce an under-relaxation coefficient in the flow step an important characteristic of the sail is that a large part of its borders is free (*i.e.* their displacements are unknown). Usually, on a main-sail for example, the displacements are fixed at the mast and the boom (where deformations are neglected), but even if strains are small, important displacements can be observed at the trailing edge . This phenomenon is much worst on jib. The assumption of small displacements then becomes erroneous and a linear approximation leads to unrealistic results.

3. A large displacement model

As observed at the end of the preceding section, linear computations are not efficient in realistic situations: nonlinear models must be introduced in order to describe large elastic displacements of the sail. Muttin (Muttin, 1989) has proposed that the behaviour of a sail be approached by considering a sequence of infinitesimal displacements of a membrane. In this approach, a problem of linear elasticity (analogous to the second level of the previous method) is to be solved at each step. The infinitesimal deformations are superimposed in order to compute a finite deformation. However, such an approach can only take into account deformations where a variation of the internal energy occurs: this is a severe restriction, since deformations of a sail are possible without variation of internal energy.

We present in the following an alternative nonlinear model for finite displacements of the sail, where the structure is considered as a network of stress-unilateral strings. This model describes deformations of the structure, for both cases of variation of the internal energy or constant internal energy: elastic or inextensible deformations are furnished by this model. So, this formulation takes into account the two main aspects of the behaviour of the sail. This formulation has been developed for fabrics and takes into account the possibility of folds in the configuration but neglects bending stresses. It leads to specific difficulties connected to the formulation of a non convex variational problem (Principle of Virtual Works). The difficulties can be functionally solved by the method of *convexification* or relaxation, that leads to a numerical method: we present a variational formulation for the relaxed problem and an algorithm for the numerical resolution.

3.1. THE STRESS-UNILATERAL PROPERTY OF STRINGS

A string is an ideally flexible structure which is not capable of transmitting compression. Under negative stress, a string modifies its geometry in such way that this condition is satisfied. Thus the energy of deformation of the string does not take in account the curvature of the fibres.

We denote by ϵ the unitary strain, by T the tension and by $K > 0$ the modulus of elasticity. Then the constitutive law is:

$$\epsilon = \frac{T}{K} \qquad \text{and} \qquad T \geq 0$$

This inequality introduces functional and numerical difficulties in the analysis of strings: non existence of equilibrium and convergence to approximated solutions which are not solution (local minimum). A Non Convex Optimisation method is used to yield a numerical method.

3.2. STRESSES IN A NETWORK OF STRINGS

We consider a network consisting of two families of strings: the "Horizontal" and "Vertical" strings which approach the sail in a natural configuration (*i.e. without strain*). Each string is defined by a constant of elasticity and a length h^s. We assume that h^s is small compared to a characteristic length of the sail.

The network is described by curvilinear coordinates $a = (a^1, a^2)$; Each line of coordinate a^i corresponds to stress-unilateral strings: the internal efforts are the tension $\vec{T} = \vec{T}_1 + \vec{T}_2$ and are given by:

$$(1) \qquad \begin{cases} \epsilon_i = \|\partial_i \vec{x}\| - 1 \\ \vec{T}_i = T_i \vec{t}_i \ , \quad \vec{t}_i = \dfrac{\partial_i \vec{x}}{|\partial_i \vec{x}|} \ , \ (i = 1, 2) \\ T_i = K_i \epsilon_i \ , \ T_i \geq 0 \ , \ (i = 1, 2). \end{cases}$$

without summation on indices.

This definition of the internal stresses indicates that only variations of length in given directions (the strings directions) induce tension in the medium. An important consequence of this tension model is that changes in relative orientation of the two strings families (angle between $\vec{t}_1$ and $\vec{t}_2$) does not lead to elastic energy variation: shear stresses are neglected too.

The field of tensions generated by the current configuration $\vec{x}$ of the sail is contained in the plane tangential to the surface and referred as $\vec{T}$. Then the energy of elastic strain is:

$$W(\vec{x}) = \int_\Omega \left[\frac{K_1}{2} (|\partial_1 \vec{x}(a)| - 1)^2 + \frac{K_2}{2} (|\partial_2 \vec{x}(a)| - 1)^2 \right] da$$

We can observe that this model for internal stresses consists in considering only the strains of the strings and then, shear stresses are neglected although they actually appear because fabrics are weave and coated. One may remarks that these stresses should be taken into account but sails' designers try as much as possible to align the directions of the strings of higher elastic moduli with internal stresses directions. As a consequence shearing modes of deformation are minimised. A more convenient modelisation of the problem would involve a higher order approximation for the Young modulus and the Poisson coefficient. However, a simpler model can be obtained by just fitting to the actual sail cut plan. An example of such a mesh is provided on figure 2. Every triangular element of the mesh has 2 sides aligned with the local strings' directions of the constitutive fabrics.

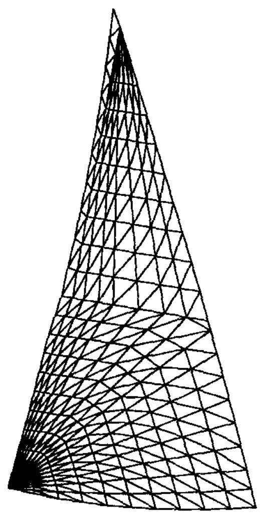

Fig. 2. – Example of a mesh for a complex panelling of the sail.

3.3. The equilibrium equations

The sail is submitted to external forces of Lagrangian densities $\vec{f} = \delta p\vec{n}$ (the pressure jump through the sail which applies a normal load). Then the equation describing the equilibrium of the structure can be written as:

$$\partial_1 \vec{T}_1 + \partial_2 \vec{T}_2 + \vec{f} = \vec{0}, \quad a \in \Omega$$

We consider the case of a sail which has fixed borders Γ_0 and other parts $\Gamma = \partial\Omega - \Gamma_0$ allowed to move. Thus, the boundary conditions read as:

$$\begin{cases} \vec{x}(a) = \vec{x}_0(a) \ , \ a \in \Gamma_0 \\ \nu_1 \vec{T}_1 + \nu_2 \vec{T}_2 = \vec{h} \ on \ \Gamma \end{cases}$$

Here, $\vec{\nu} = (\nu_1, \nu_2) \in R^2$ denotes the outwards unit normal to Ω. We observe that the first condition corresponds to a rigid link between the sail and the boat (by means of the mast and/or boom considered as rigid). Some authors consider that this condition must be modified in order to take into account an elastic link (see for example (Muttin, 1989)). Such a generalisation is immediate by using the second relation ($\vec{h}$ depends on the elastic link) and does not involve new theoretical difficulties. In the following, we will restrain ourselves to the rigid case ($\vec{h} = \vec{0}$).

3.4. A variational formulation (principle of virtual works)

We denote by V_0 the set of admissible displacements (null on Γ_0):

$$V_0 = \{\vec{y} : \Omega \longrightarrow R^3 \mid \vec{y} = \vec{0} \ on \ \Gamma_0\}.$$

Let us introduce:

$$\begin{cases} \left(\vec{T}, [\vec{u}_1, \vec{u}_2]\right) = \int_\Omega \left(\vec{T}_1.\vec{u}_1 + \vec{T}_2.\vec{u}_2\right) da \\ F\left(\left[\vec{c}, \vec{h}\right], \vec{y}\right) = \int_\Omega \vec{c}.\vec{y} da + \int_\Gamma \vec{h}.\vec{y} d\Gamma \end{cases}$$

Then a formulation of the steady equilibrium is as follows:

$$\begin{cases} \vec{x} \in V = \vec{x}_0 + V_0 \\ \left(\vec{T}, \left[\partial_1 \vec{y}, \partial_2 \vec{y}\right]\right) = F\left(\left[\vec{f}, \vec{h}\right], \vec{y}\right), \forall \vec{y} \in V_0 \end{cases}$$

3.5. Functional resolution

The meaning of the previous equation is that $\vec{x}$ is a configuration solution of our problem if and only if $\forall \vec{y}$ in the set of the admissible displacement, the displacement field $\vec{y}$ induces a positive variation of the energy of the total system (*i.e.* that for such a displacement from $\vec{x}$, variation of elastic deformations energy is positive). Moreover, $\vec{x}$, as we mentioned above, has to satisfies some additional constrains:

1. $\vec{x}(a) = \vec{x}_0(a) \ on \ \Gamma_0$ (boundary condition)
2. $\|\partial_\alpha \vec{x}\| \geq 1$ (unilateral stress behaviour)

As a consequence of the unilateral stress behaviour, the set of admissible configurations is nonconvex and it leads to numerical difficulties. We introduce as K_{in} the set of admissible configurations which satisfies the previous conditions:

$$K_{in} = \{\vec{x} \in V_0 \mid \|\partial_\alpha \vec{x}\| \geq 1 \ on \ \Omega\}$$

Let us introduce $\alpha^+ = \dfrac{(\alpha + |\alpha|)}{2}$, the total energy $J = W + U$ and the relaxed energy $J^{**} = W^{**} + U$, given by:

$$\begin{cases} W^{**}(\vec{x}) = \int_\Omega \left\{ \frac{K_1}{2} \left[(\|\partial_1 \vec{x}\| - 1)^+\right]^2 + \frac{K_2}{2} \left[(\|\partial_2 \vec{x}\| - 1)^+\right]^2 \right\} da \\ U(\vec{x}) = -F([\vec{g}, \vec{h}], \vec{y}) \end{cases}$$

For this particular functional, $J^{**} = QJ = \overline{J}$, where QJ is the quasi-convex regularization of J and $\overline{J}$ is its lower semi-continuous regularisation (see (Dacorogna, 1989)). Thus we consider the following problems:

Problem 1: Find the configuration $\vec{x} \in K_{in}$ which satisfies (1) and:

$$\begin{cases} \nu_1 \vec{T}_1 + \nu_2 \vec{T}_2 = \vec{h} \ on \ \Gamma \\ \left(\vec{T}, \left[\partial_1 \vec{y}, \partial_2 \vec{y}\right]\right) = F\left(\left[\vec{f}, \vec{h}\right], \vec{y}\right), \forall \vec{y} \in V_0 \end{cases}$$

Problem 2: Find the configuration $\vec{x} \in K_{in}$ such that $J(\vec{x}) = inf_{K_{in}}\{J\}$

Problem 3: Find the configuration $\vec{x} \in V_0$ such that $J^{**}(\vec{x}) = inf_{V_0}\{J^{**}\}$

A mathematical analysis of these problems leads to the following results (see (Souza de Cursi, 1987)):

(1) A configuration $\vec{x}$ corresponds to a solution of the problem 1 **if and only if** it is solution of the problem 2

(2) $\vec{x}$ is a solution of the problem 2 **if and only if** $\vec{x}$ is a solution of the problem 3 **and** $\vec{x} \in K_{in}$

(3) The field of tensions $\vec{T}$ is uniquely determined and is the same for the problems 1, 2 and 3

(4) $\vec{x}$ is a solution of problem 2 **if and only if** $\vec{x} \in K_{in}$ and $\vec{x}$ generates the field of tensions $\vec{T}$

(5) The problem 3 has at least one solution.

3.6. A numerical method for the string network

The previous results give us a numerical method for the prediction of the equilibrium shape of a sail under an external field of forces. Problem 1 describes the behaviour of a sail, under the assumption mentioned above and for the approximation of the material by a strings network.

We solve this problem in two steps:

Step 1: we solve the Problem 3 which gives us the field of tensions $\vec{T}$ which is also solution of Problem 1.

Step 2: we look for an admissible configuration $\vec{x} \in K_{in}$ which generates the previous field of tensions ; this configuration is a configuration of equilibrium and is unique if the strain is greater than zero for every string.

Thus we shall establish a numerical method for the problem 3.

3.7. Reduction to a variational equation

Since problem 3 is a convex differentiable problem, it is equivalent to the following variational problem:

$$\begin{cases} \vec{x} \in V = \vec{x_0} + V_0 \\ a(\vec{x}, \vec{y}) = F([\vec{f}(\vec{x}), \vec{h}], \vec{y}) \ , \ \forall \vec{y} \in V_0 \end{cases}$$

Where

$$a(\vec{x}, \vec{y}) = a_1(\vec{x}, \vec{y}) + a_2(\vec{x}, \vec{y})$$

$$a_\alpha = \int_\Omega K_\alpha (|\partial_\alpha \vec{x}| - 1)^+ \frac{\partial_\alpha \vec{x}}{|\partial_\alpha \vec{x}|} . \partial_\alpha \vec{y} da$$

The problem can be solved using a Finite Element Method (FEM. See (Zienkiewicz, 1977)).

3.8. Application to the problem of a sail

We consider the situation of a jib (the case of a main-sail differs on the number of fixed borders Γ_0). The sail-maker provides a geometry for a natural configuration (the strain are null everywhere).

Thus we set on the surface $(M+1).(N+1)$ constitutive points of a mesh made of V_i $(1 \leq i \leq M+1)$ "Vertical" per H_j $(1 \leq j \leq N+1)$ "Horizontal" curved lines supposed to be aligned with the strings that approximate the material. Γ_0 corresponds to V_1 (the luff) and H_{N+1} (the header); the intersection of V_i and H_j is the point P_{ij}. With the diagonals $P_{i+1,j} - P_{i,j+1}$ we built $2.M.N$ triangles where the unknowns are

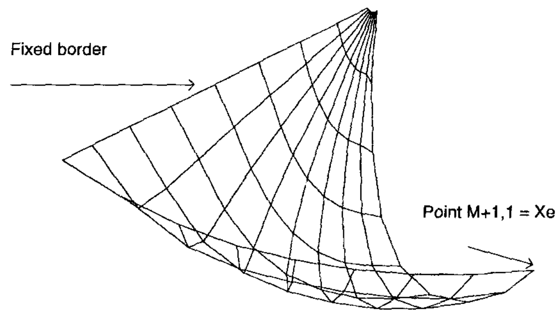

Fig. 3. – Sketch of the mesh.

approximated by polynomials of degree 1. Let $\vec{X}_{ij}$ be the approximated value of $\vec{x}(P_{ij})$ so the unknowns of the problem are $\chi = (\vec{X}_{ij})_{2\leq i\leq M+1, 1\leq j\leq N}$. A view of a mesh is presented on figure 5.

A flow computation is then performed using a particles method (see (Charvet, 1992b), (Charvet, 1992a)) and the inextensible model. This computation provides the external efforts $\vec{f}$ for the guess configuration. The approximated value of $\vec{f}(\vec{x})$ at the point $\vec{X}_{i,j}$ is $\vec{f}_{i,j}$.

In order to take into account the mobility of the third extremity of the sail (the tack) (which corresponds to the point $\vec{X}_{M+1,1}$) where the jib is tied, a penalty method is used on the distance from this point to a fixed point of the deck (the length of the sheet):

$$\vec{f^*}_{M+1,1} = \vec{f}_{M+1,1} - \frac{\vec{X}_R - \vec{X}_{M+1,1}}{\|\vec{X}_{M+1,1} - \vec{X}_R\|}(D_{JR} - \|\vec{X}_{M+1,1} - \vec{X}_R\|)C_{pen}$$

Where $\vec{X}_R$ is a fixed point on the deck (the jib roller), C_{pen} is a penalty coefficient and D_{JR} is the length of the sheet. In the following, the tack $\vec{X}_{M+1,1}$ will be denoted by $\vec{X}_e$.

The variational equation of Problem 3 is then approached using a standard Finite Element Method which leads to a nonlinear system of equations for the unknowns:

$$\Xi(\chi) = \vec{0}\ ;\ \Xi(\chi) = (\vec{E}_{ij}(\chi))_{2\leq i\leq M+1, 1\leq j\leq N+1}$$

This system is solved by an iterative procedure: from the initial guess configuration, we compute $\chi^{(1)}$, $\chi^{(2)}$,..., by:

$$\vec{X}_{ij}^{k+1} = \vec{X}_{ij}^{k} - \omega\vec{E}_{ij}(\vec{X}_{ij}^{k})$$

Where ω is an under-relaxation coefficient. The quality of the approximated solution depends on this coefficient and can be controlled by the mean value residue:

$$R_2 = \left(\sum_{i=2}^{M}\sum_{j=1}^{N}\|\vec{E}_{ij}(\chi^{(k)})\|^2 + \sum_{j=2}^{N}\|\vec{E}_{M+1,j}(\chi^{(k)})\|^2\right)^{\frac{1}{2}}$$

4. Numerical results

The characteristics conditions of the flow are (in the assumption of in-viscid fluid) the free stream velocity $\vec{U}_\infty$, a typical length of the sail L_s and the density of the fluid ρ. Then we build the characteristic pressure $P_c = \rho\|\vec{U}\|^2/2$ and force $F_c = \rho\|\vec{U}\|^2 L_s^2/2$ to consider the non dimensional problem. The mechanical characteristics of the sail are $K_c = K/P_c$ equal for the two families of strings and $h_c = h/L_s$, where K is the modulus of elasticity of the material and h the thickness of the sail.

4.1. Definition of the reference sail

For the computations, we have selected the following values:

1: the velocity of the free stream $\vec{U}_\infty$ is taken equal to 10 $knots$.

2: the angle between the boat axis and $\vec{U}_\infty$ is 20 $degrees$.

3: the angle between the boat axis and the bottom of the sail is 7 $degrees$.

4: the characteristic length is the length of the luff which is equal to 9.4 m and the thickness of the sail is 0.5 mm.

5: the modulus of elasticity of the strings is 5 MPa.

Then a flow computation is performed using the first level model to find out the equilibrium geometry in the inextensible assumption. Hereafter, the sail-maker guess geometry will be referred to as the "natural geometry", whereas the modified equilibrium shape which account for aerodynamic loads only will be referred to as the "initial geometry". This last one is presented in figures 4 and 5.

We emphasise that this geometry is a natural configuration (strains are null). The external loads obtained are then supposed to be independent of the current geometries. *i.e.* that the interaction with the flow will not be considered in the following, which is obviously an important simplification of the problem, but we only get interested here in the study of the behaviour of the strings network model.

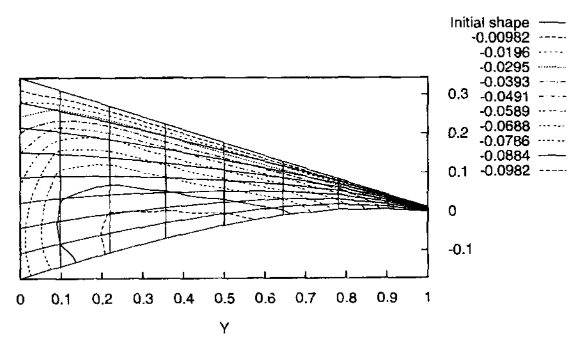

Fig. 4. – Initial geometry - Iso-Z lines

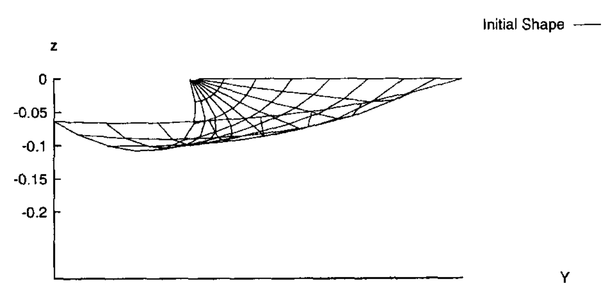

Fig. 5. – Top view of initial geometry.

4.2. Computation with fixed tack

We performed a first computation assuming that the point $\vec{X}_e$ is fixed. This computation gives us a prediction of the deformations due to the elasticity for a wind rising from 0 knots to 10 knots (the characteristic velocity $\vec{U}_\infty$) without any change of the boundary conditions. Since the initial geometry does not present any initial strain, the deformations are quite important but seem to be realistic if we consider that the interaction with the flow is not taken into account. Nevertheless, it puts in evidence the limit of linear models when boundary conditions lead to actual large displacement simply by elastic effect as plotted in figures 6 and 7.

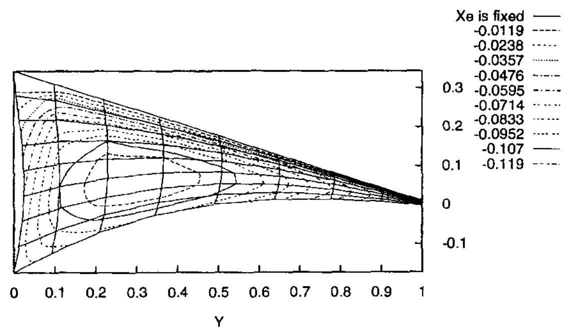

Fig. 6. – Final shape for Xe fixed.

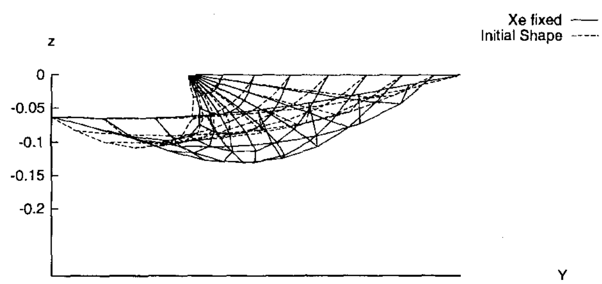

Fig. 7. – Top view of final shape for Xe fixed.

4.3. Fixed bottom of the sail

4.3.1. *Deformation*

In order to obtain realistic results, Charvet (Charvet, 1992a) had to fix the bottom of the sail for the second level of his model. To quantify the effect of fixing this border on the displacement field, we have performed this case. It appears that even if the displacement of this border is small, its influence on the solution everywhere is important for the strings network model considered. The results are presented in figures 8, 9 and 10.

The displacement field from the initial geometry to the final shape in the case of the fixed bottom shows small values such that the assumption of small displacement is valid, but this assumption is inconsistent for the case where only $\vec{X}_e$ is fixed. This result may explain the unrealistic results obtained by Charvet when he allows this border to move. This is an evidence of the general improvement born by the nonlinear model.

4.3.2. *Convergence of the iterative procedure*

On figure 11, we present the evolution of the mean value residue R_2 (defined in the end of subsection 3.8) during the iterative procedure used to solve the nonlinear system of equations of the problem and for different

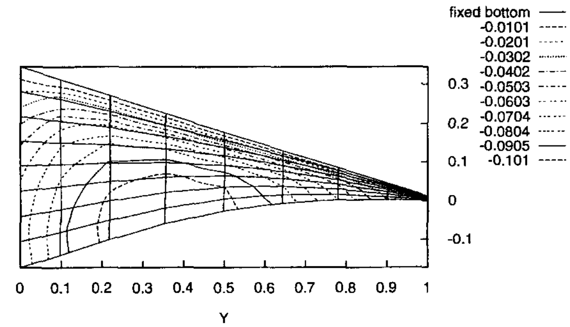

Fig. 8. – Final shape for fixed bottom.

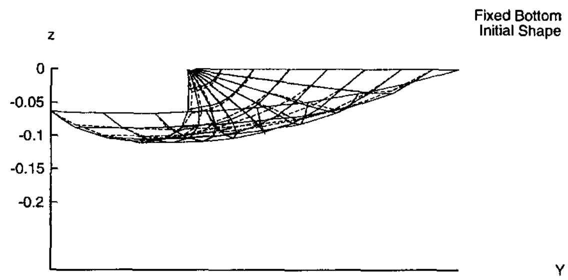

Fig. 9. – Top view of final shape for fixed bottom.

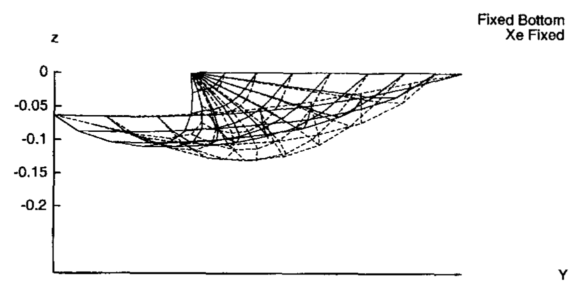

Fig. 10. – Comparison of final shape for free or fixed bottom.

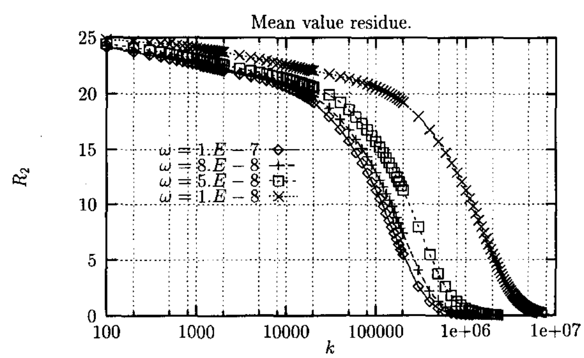

Fig. 11. – Evolution of the mean value residue R_2 versus number of iteration k for different values of the under-relaxation coefficient ω. Computations are stopped when R_2 becomes smaller than $1.E - 3$ or when k becomes greater than $1.E7$.

values of the under-relaxation coefficient ω. For these computations, the iterative scheme is applied as long as both the following conditions are satisfied:

$$R_2 \geq 0.001$$

$$k \leq 1.E7$$

For every values of ω presented, R_2 has the same behaviour and the log diagrams put in evidence a three steps evolution. Firstly, during the beginning of the process (from $k = 1$ to $k \approx 1.E4 - 5.E - 4$), R_2 slowly decreases with $\log k$ and the slopes depend on the value of ω. This stage corresponds to the iterations where the configuration is deformed without significant changes of the elastic energy $W^{**}(\vec{x})$: the sail primarily adapts itself to the field of external loads in the way close to that described in first level of Charvet's model (inextensible stage) (see also figure 12 for the evolution of the elastic energy W^{**} with iteration number k).

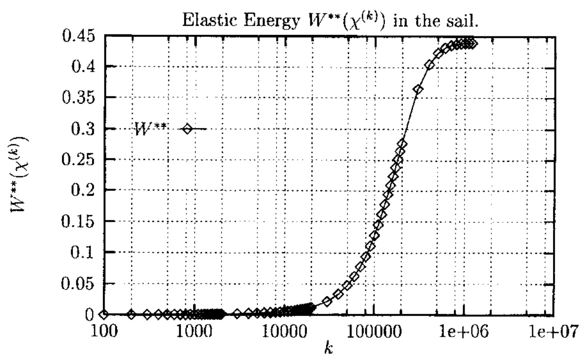

Fig. 12. – Evolution of the elastic energy W^{**} contained in the sail during the deformation process and with $\omega = 1.E - 7$ and a fixed bottom.

Then a transition appears and R_2 quickly decreases with $\log k$ (while W^{**} increases on figure 12). This period corresponds to the apparition of significant strains: the displacement from the initial configuration (which is natural) then induces changes in the natural lengths of the strings leading to elastic tensions. This second stage has a constant slope and lasts till the solution is converged to a value which depends on the choice of the relaxation parameter.

Then the third step presents a very low convergence rate for R_2 to a value which is as small as ω is small. During this low slope convergence period, the displacement of the configuration is negligible as illustrated of figure 12 where the elastic energy is constant.

4.3.3. *Computational time*

Table 4.3.3 presents the computational times needed to obtain a solution with an accuracy measured by $R_2 \leq 1.10^{-3}$. We have tested the previous values of ω. For $\omega > 10^{-7}$ the iterative procedure quickly diverges since large values of ω allow large displacements from an iteration to another and then induces sudden large changes in strains. For very small values of ω, the iterative procedure stills converge but computational time increases very quickly while significant improvement in accuracy can not be expected.

4.4. Influence of the length of the sheet

We set $\vec{X}_R$ at $(-0.25, -0.8, -0.63)$ and D_{jR} at 0.112 (which is the length of the sheet for $\vec{X}_e$ fixed in the initial geometry). We have performed a first computation without changing the value of D_{jR} but $\vec{X}_e$ is

TABLE 4.3.3. – computational times for a fixed bottom sail and different values of ω. The table also indicates the number of iterations necessary to achieve an accuracy of $R_2 \leq 10^{-3}$. For $\omega = 1.10^{-8}$ the accuracy is not achieved after 7 000 000 iterations when the computation is stopped.

ω	Number of iterations	Accuracy	Time (HP 712/100)
1.10^{-7}	1 249 014	$R_2 \leq 10^{-3}$	7 min. 15s.
8.10^{-8}	1 577 245	$R_2 \leq 10^{-3}$	9 min. 03s.
5.10^{-8}	2 319 967	$R_2 \leq 10^{-3}$	13 min. 52s.
1.10^{-8}	7 000 000	$R_2 = 1.998 \quad 10^{-1}$	42 min. 12s.

now allowed to move to its position of equilibrium. The results are plotted on figure 13. We notice that the configuration of equilibrium is different from that of the case with $\vec{X}_e$. We emphasise on the limit of the linear models to deal with this kind of problems for which the boundaries position is unknown except for the luff, leading to large displacements even for small loads perturbations.

Then we impose $D_{jR} = 0.082$. In figure 14, the final geometry is plotted. This configuration is flatter than the initial shape. This value of D_{jR} is such that strains remain even if external loads are null. This result put in evidence one of the advantage of the nonlinear model: to compute an equilibrium configuration, considering the elastic deformations, it just necessitates to know one natural configuration of the sail (*id est* without strain). This advantage has many applications for the sail makers because contrary to the small displacement assumption it does not need a guessed configuration. However, we can understand that small perturbations of the external loads would not induce large displacements since $D_{jr} = 0.082$ is such that elastic energy can not be zero. On the contrary, for the case $D_{jr} = 0.112$, it exists a large set of zero strain configurations and then many large displacement fields for which the elastic energy is constant.

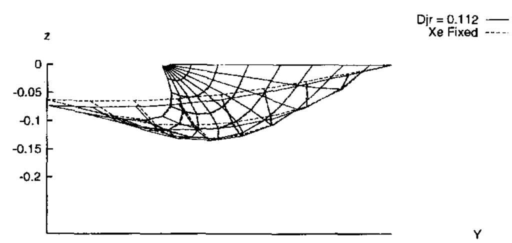

Fig. 13. – Top view of final shape for $Djr = 0.112$.

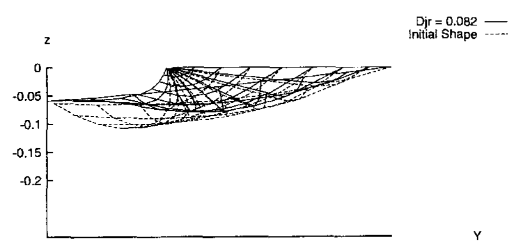

Fig. 14. – Top view of final shape for $Djr = 0.082$.

4.5. CONVERGENCE TEST

From the initial geometry, we have computed the equilibrium configuration for the opposite field of external loads in reference of the plane defined by the luff and the vector which links the clew to the roller jib. This computation case, defined by Charvet as a 'tack', proves the great capabilities of the strings network model which can support a displacement field which does not change the internal energy and find the equilibrium configuration even if the initial (given) shape is far from the solution. A few intermediate configurations, extracted from the iterative procedure introduced in section (3.8), are plotted in figures 15.a to 15.f. We emphasise that this computation has been done considering the elastic deformations. So the intermediate solutions are not necessary admissible configurations in the sense of positive strain everywhere ($\vec{x} \in K_{in}$). Obviously, the final shape (figure 15.f) is an equilibrium configuration and satisfies the constrains of the strings network representation.

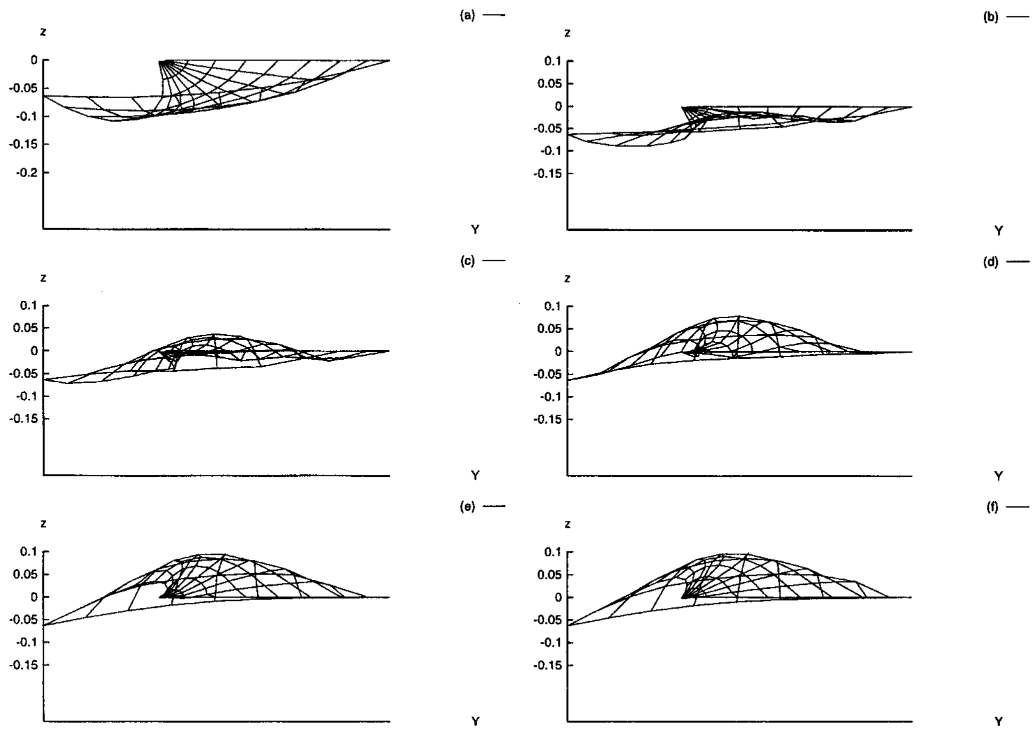

Fig. 15. – Intermediate configurations (a-e) and final shape (f) for convergence test.

5. Conclusions

5.1. Connection of the two finite displacement models

The variational equation used to solve the problem 3 of the finite displacement model, can be read in the following way: Find the configuration $\vec{x}$ which minimise the mechanical energy of the sail $J^{**}(\vec{x})$. Thus, we have to minimise the sum of $U(\vec{x})$ and $W^{**}(\vec{x})$.

We can notice that $U(\vec{x}) = -F([\vec{f}, \vec{h}], \vec{x})$ in its discretized form is equivalent to the functional of the inextensible model since $\vec{h} = 0$ and $\vec{f} = \vec{n}\delta P$. So the relaxed elastic energy $W^{**}(\vec{x})$ appears to be a penalization on the elastic deformation added to the inextensible problem. Then the two models are equivalent for an infinite penalization of the strains. The modulus of elasticity K can be considered as a penalty coefficient on strains introduced in the functional.

In fact, on the one hand we have for $\lim_{\infty} K$ the elastic problem becomes: *find* $\vec{x}$ *which minimise* $J^{**} = U(\vec{x}) = -F([\vec{f}, \vec{h}], \vec{x})$ *and* $||\partial_\alpha \vec{x}|| = 1$ (which is equivalent to the conservation of geodesic distances). And on the other hand, the constrain of inextensibility leads to $W^{**} = 0$.

5.2. The unsteady problem

Considering the previous numerical results, the strings network model seems to be adapted to predict the behaviour of sails under unsteady conditions due to unsteady inflows (variations of the wind direction or gusts) and/or due to the motion of the boat. If the inertia of the fabric can be neglected (*i.e.* for very light materials or large time scales of deformation), the problem of steady equilibrium presented above has to be solved at every time step, according to the instantaneous external loads estimated with the actual shape of the sail. On the contrary, when the effects of inertia of sails and the external loads are of the same order of magnitude, the quasi-steady approach of the deformation is no longer valid. It is then necessary to introduce the kinetic energy of the sail in the functional of the problem and to use a time scheme to estimate velocities and accelerations for the configuration. Moreover the nonlinearity of the model (and of the flow) causes theoretical difficulties in the use of an implicit time scheme and will certainly require a variational formulation and internal cycles similar to those used for the steady strings network problem.

5.3. Future developments

The future developments of the sail models, for the structure point of view, should take into account reinforced areas and "rigidifying" devices to fit as much as possible the real sail. The study of the influence of the mesh size on the solution for both assumptions of elastic or inextensible behaviour seems to be necessary too. As a matter of fact, the two levels method should be included in an iterative procedure which would lead to a method of the same kind as the unified approach used in the strings network model. Moreover it would provide an insight on stability of the computation and offer the possibility to compare the results of the two models.

The model presented considers the sail as a network of strings. So this model neglects shear and bending effects even if it reproduces these deformation modes. It leads to an underestimation of the elastic energy in the structure and in order to obtain a more realistic model, the total energy of the structure must be modified by the inclusion of shear and bending terms. Thus, more theoretical and numerical works are necessary and will allow to have a better estimation of the elastic energy influence on the steady equilibrium configuration.

All actual and future models should be improved on real cases that will require experimental data and measurements in ideal or real sailing conditions. Unfortunately, such information is not available or is considered confidential, so experimental campaigns should be planned.

The authors thank F. Hauville who provided the reference configuration and performed the flow computation, and the Regional Council of Haute Normandie for its financial support.

REFERENCES

ANTMAN S. S., 1995, *Nonlinear problems of elasticity*, **107**, Springer-Verlag.
CHARVET T., 1992a, *Résolution numérique de problèmes liés au comportement des voiles de bateau*, PhD thesis, Ecole Polytechnique, Paris.
CHARVET T., HUBERSON S. G., 1992b, Numerical calculation of the flow around sails, *European Journal of Mechanics*, B/Fluids **11**, 599-610.
CIARLET P. G., 1978, *The Finite Element Method for Elliptic Problems*, North Holland.
DACORAGNA B., 1989, *Direct methods in the Calculus of Variations*, Springer Verlag.
DESTUYNDER P., 1990, *Modélisation des coques minces élastiques*, Masson, Paris.

FUKASAWA T., KATORI M., 1992, Numerical approach to aeroelastic responses of three-dimensional flexible sails. In *The Eleventh Chesapeake Sailing Yacht Symposium.*

JACKSON P. S., 1985, The analysis of the three-dimensional sails, *Proceedings of the 10th Canadian Congress of Applied Mechanics*, Univ. of Western Ontario, London, Canada, 59-67.

JACKSON P. S., CHRISTIE G. W., 1986, Numerical analysis of three-dimensional elastic membrane wings, *AIAA Journal*, **25**, 5, 676-682.

LE DRET H., RAOULT A., 1995, The nonlinear membrane model as variational limit of nonlinear three-dimensional elasticity, *J. Math. Pures Appl.*, **74**, 549-578.

MUTTIN F., 1989, *Modélisation de membrane. Application à l'analyse mécanique des voiles de bateau*, PhD thesis, Université de Nice Sophia Antipolis.

SOUZA DE CURSI J. E., 1987, Statique des fils sans raideur à comportement unilatéral, *J. Méc. Th. Appl.*, **6(6)**, 789-825.

ZIENKIEWICZ O. C., 1977, *The Finite Element Method in Engineering Sciences*, Mc Graw Hill, 1977.

(Manuscript received February 22, 1996;
Revised October 03, 1997.)