The 3é hyperelastic model applied to the modeling of 3D impact problems

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Abstract

Lainé et al. [Nonlinear isotropic constitutive laws: choice of the three invariants, convex potentials and constitutive inequalities, Int. J. Eng. Sci. 37 (1999) 1927–1941.] have proposed a new third order hyperelastic model, named here the 3é model. The present paper is devoted to the modeling of finite deformations of hyperelastic bodies described by the 3é model under contact/impact conditions. A total Lagrangian formulation is adopted to describe the geometrically nonlinear behavior. A first order algorithm is applied to integrate the equations of motion. For the finite element implementation, an explicit expression of the tangent operator is derived. Efficiency and accuracy of the resulting method are illustrated on a numerical example.

Keywords: Hyperelasticity; Contact/impact; Large deformation

1. Introduction

Problems involving contact and friction are among the most difficult ones to solve in mechanics and at the same time of practical importance in many engineering branches. A large number of algorithms for the modeling of contact problems by the finite element method have been presented in the literature. See for example the monographs by Kikuchi and Oden [1], Zhong [2], Wriggers [3], Laursen [4] and the references therein. de Saxcé and Feng [5] have proposed a bi-potential method combined with an augmented Lagrangian formulation. Feng et al. [6] have successfully applied this method to the modeling of static contact problems between Blatz–Ko hyperelastic bodies.

Regarding the time integration for implicit dynamic analysis in structural mechanics, the most commonly used scheme is the second order scheme such as Newmark, Wilson-∫, HHT [7]. A first order time scheme has also been proposed by Jean [8] for time stepping in granular mechanics. Recently, Feng et al. [9] have applied this scheme for the modeling of impact problems between elastic bodies.

In nonlinear elasticity, there exist many constitutive models describing the hyperelastic behavior of foam-like or rubber-like materials [10–16]. These models are available in many modern commercial finite element codes. In 1999, Lainé, Vallée and Fortuné [17] have proposed a new third order hyperelastic model, referred here as the 3é model. The aim of the present paper is to propose a finite element implementation of this model, in view of application to impact problems involving large displacements and large strains. A numerical example is presented in this study to show the validity of the model developed. This example consists of the impact of two hyperelastic blocks. Both frictionless and frictional contacts are considered to highlight the physical energy dissipation by frictional effects.

2. Hyperelastic bodies and the 3é model

Rubber-like materials are usually taken to be hyperelastic and often undergo large deformations. To describe the geometrical transformations in \( \mathbb{R}^3 \), the deformation gradient tensor is introduced by

\[
F = I + \nabla u = I + \frac{\partial u}{\partial x}, \tag{1}
\]
The eigenvalues of stress and strain tensors are sorted in the same order: the eigenvector associated to the highest eigenvalue of one of the strain tensors and whose derivative with respect to one strain component determines the corresponding stress component. This can be expressed by

$$E = \frac{1}{2} (C - I), \quad E_d = E - \frac{1}{3} \text{tr}(E) \cdot I.$$  

(3)

For an hyperelastic law, an elastic potential function $W$ exists (or strain energy density function) which is a scalar function of one of the strain tensors and whose derivative with respect to one strain component determines the corresponding stress component. This can be expressed by

$$S = \frac{\partial W}{\partial E} = \frac{\partial W}{\partial C}.$$  

(4)

where $S$ is the second Piola–Kirchoff stress tensor. In particular, for isotropic hyperelasticity [12], Eq. (4) can be written as

$$S = 2 \left[ I_1 \frac{\partial W}{\partial \text{tr} I_1} C^{-1} + \left( \frac{\partial W}{\partial \text{tr} I_2} \right) I - \frac{\partial W}{\partial \text{tr} I_3} C \right].$$  

(5)

where $I_i$ ($i = 1, 2, 3$) denote the invariants of the right Cauchy–Green deformation tensor $C$:

$$I_1 = \text{tr}(C), \quad I_2 = (I_1^2 - C : C)/2, \quad I_3 = \text{det}(C) = J^2.$$  

(6)

The constitutive law, proposed by Lainé, Vallée and Fortuné in 1999 [17], describes isotropic compressible or incompressible rubber-like materials. This model is based on the following main ideas:

- If the constitutive law relating $E$ and $S$ is isotropic, the strain tensor $E$ has the same eigenvectors as the stress tensor $S$.
- The eigenvalues of stress and strain tensors are sorted in the same order: the eigenvector associated to the highest eigenvalue of the strain tensor is also associated to the highest eigenvalue of the stress tensor.

New invariants of $S$: $(X, Y, Z)$ and of $E$: $(x, y, z)$ are thus introduced as follows

$$X = \sqrt{\text{tr}(E_d)^2 \cos \varphi}, \quad Y = \sqrt{\text{tr}(S_d)^2 \sin \varphi}, \quad Z = \frac{\text{tr}(S)}{\sqrt{3}}.$$  

(7)

$$x = \sqrt{\text{tr}(E_d)^2 \cos \varphi}, \quad y = \sqrt{\text{tr}(E_d)^2 \sin \varphi}, \quad z = \frac{\text{tr}(E)}{\sqrt{3}}.$$  

(8)

The so-called Lode’s angles (ranging from 0 to $\pi/3$) are defined by

$$\psi = \frac{1}{3} \cos^{-1} \left[ \frac{3\sqrt{6}}{\text{det}(S_d)} \left( \frac{\text{tr}(S_d)}{\text{tr}(S_d)^2})^{1/2} \right) \right],$$

$$\varphi = \frac{1}{3} \cos^{-1} \left[ \frac{3\sqrt{6}}{\text{det}(E_d)} \left( \frac{\text{tr}(E_d)}{\text{tr}(E_d)^2})^{1/2} \right) \right].$$  

(9)

The choice of the invariants is such that

$$\text{tr}(E) = x + y + z = E_1 S_1 + E_2 S_2 + E_3 S_3,$$  

(10)

where $E_i$ and $S_i$ ($i = 1, 2, 3$) stand, respectively, for the eigenvalues of $E$ and $S$. If the eigenvalues are sorted in decreasing order, one obtains by means of the Cardan’s formula

$$S_1 = \frac{\text{tr}(S)}{3} + \frac{2}{3} \sqrt{\text{tr}(S_d)^2 \cos \psi},$$

$$S_2 = \frac{\text{tr}(S)}{3} + \frac{2}{3} \sqrt{\text{tr}(S_d)^2 \cos \left( \frac{2\pi}{3} - \psi \right)},$$

$$S_3 = \frac{\text{tr}(S)}{3} + \frac{2}{3} \sqrt{\text{tr}(S_d)^2 \cos \left( \frac{2\pi}{3} + \psi \right)},$$  

(11)

$$E_1 = \frac{\text{tr}(E)}{3} + \frac{2}{3} \sqrt{\text{tr}(E_d)^2 \cos \varphi},$$

$$E_2 = \frac{\text{tr}(E)}{3} + \frac{2}{3} \sqrt{\text{tr}(E_d)^2 \cos \left( \frac{2\pi}{3} - \varphi \right)},$$

$$E_3 = \frac{\text{tr}(E)}{3} + \frac{2}{3} \sqrt{\text{tr}(E_d)^2 \cos \left( \frac{2\pi}{3} + \varphi \right)}.$$  

(12)

In view of Eqs. (7), (8), (11) and (12), we obtain a linear relationship between the eigenvalues of $S$ (resp. $E$) and the invariants of $S$ (resp. $E$):

$$\left\{ \begin{array}{l}
S_1 \\
S_2 \\
S_3
\end{array} \right\} = \frac{2}{\sqrt{6}} \left[ \begin{array}{ccc}
0 & \frac{1}{\sqrt{3}} \\
-1 & \frac{1}{\sqrt{3}} \\
-1 & \frac{1}{\sqrt{3}}
\end{array} \right] \left\{ \begin{array}{l}
X \\
Y \\
Z
\end{array} \right\},$$  

(13)

$$\left\{ \begin{array}{l}
E_1 \\
E_2 \\
E_3
\end{array} \right\} = \frac{2}{\sqrt{6}} \left[ \begin{array}{ccc}
0 & \frac{1}{\sqrt{3}} \\
-1 & \frac{1}{\sqrt{3}} \\
-1 & \frac{1}{\sqrt{3}}
\end{array} \right] \left\{ \begin{array}{l}
x \\
y \\
z
\end{array} \right\}.$$  

The fourth order strain energy density function expressed in terms of the new invariants of the strain tensor is

$$W(x, y, z) = \left( G + \frac{a_3}{4} \varepsilon^2 z^2 \right) (x^2 + y^2) + \frac{3K}{2} z^2 + \frac{a_1}{3} (x^3 - 3xy^2) + \frac{a_2}{3} y^3 + \frac{a_4}{4} x^4 + \frac{a_5}{4} (x^2 + y^2)^2,$$  

(14)
where $G$ is the shear modulus, $K$ the bulk modulus and $a_i$ ($i = 1, \ldots, 5$) are parameters of the model. The potential is convex and differentiable if the parameters are such that
\[
a_3 \geq 0, \quad a_4 \geq 0, \quad a_5 \geq 0, \\
\sqrt{2Ga_5} \geq a_1 \geq -\sqrt{2Ga_5}, \quad 3\sqrt{Kd_4} \geq a_2 \geq -3\sqrt{Kd_4}. \tag{15}
\]

In this work, we suppose an uncoupling between spherical and distortional parts of $E$, i.e., $a_3 = 0$.

In view of Eq. (4), we have
\[
X = \frac{\partial W}{\partial x}, \quad Y = \frac{\partial W}{\partial y}, \quad Z = \frac{\partial W}{\partial z}. \tag{16}
\]

It is noted that for $a_i = 0$, we recover the well-known Neo-Hookean strain-energy density
\[
W(x, y, z) = G(x^2 + y^2) + \frac{3K}{2}z^2, \tag{17}
\]
or equivalently
\[
W(I_1, J) = \frac{G}{2}(I_1 - 3) + \frac{K}{2}(J - 1)^2. \tag{18}
\]

By deriving the energy density (14) with respect to the strain tensor, it yields
\[
S = z(3K + a_2z + a_4z^2)\frac{1}{\sqrt{3}} + \left[2G + \frac{a_5}{2}(x^2 + y^2)\right]E_d + \sqrt{a_1}\left[(E_d)^2 - \frac{x^2 + y^2}{3}I\right]. \tag{19}
\]
The Cauchy stress (or true stress) tensor $\sigma$ is calculated from the second Piola–Kirchhoff stress tensor $S$:
\[
\sigma = \frac{1}{F}FSF^T. \tag{20}
\]

To construct the tangent stiffness matrix for the analysis of nonlinear structures by the finite element method, one has to determine the stress–strain tangent operator $D$, which is a fourth order tensor resulting from the derivation of $S$ with respect to $E$ in Eq. (19):
\[
D = \frac{\partial S}{\partial E} = \left[K + \frac{2a_2}{3}tr(E) + \frac{a_4}{3}tr(E)^2\right]I \otimes I + \left[2G + \frac{a_5}{2}tr(E_d)^2\right]\left(I \otimes I - \frac{1}{3}I \otimes I\right) + a_3E_d \otimes E_d + \sqrt{a_1}\left(I \otimes E_d + E_d \otimes I\right) - \frac{2}{3}E_d \otimes I - \frac{2}{3}I \otimes E_d. \tag{21}
\]

The coordinate-free symbols $\otimes$, $\otimes$ and $\otimes$ used above are related to the corresponding index symbols in the following way:
\[
(A \otimes B)_{ijkl} = A_{ij}B_{kl}, \quad (A \otimes B)_{ijkl} = A_{ik}B_{jl},
\]
\[
(A \preceq B)_{ijkl} = \frac{1}{2}(A_{ij}B_{kl} + A_{il}B_{jk}). \tag{22}
\]

3. Finite element formulation of nonlinear structures

3.1. Total Lagrangian formulation

In the case of dynamic multibody contact problems with large deformations of hyperelastic solids, the nonlinear relationship between strains and displacements cannot be ignored. The total Lagrangian formulation is adopted in this work to describe nonlinear behavior. It is well known that the strain tensor $E$ and the stress tensor $S$ are both symmetric. Thus, we note hereafter $E$ and $S$ in vector form as
\[
E = (E_{11} \ E_{22} \ E_{33} \ 2E_{12} \ 2E_{13} \ 2E_{23})^T,
\]
\[
S = (S_{11} \ S_{22} \ S_{33} \ S_{12} \ S_{13} \ S_{23})^T. \tag{23}
\]

In the context of the finite element method and with Eqs. (1) and (5), the Green–Lagrange strain can be formally written with linear and nonlinear contributions in terms of nodal displacements [15]:
\[
\delta E = (B_L + \frac{1}{2}B_{NL}(u))\delta u, \tag{24}
\]

where $B_L$ is the matrix which relates the linear part of the strain term to the nodal displacements, and $B_{NL}(u)$, the matrix which relates the nonlinear strain term to the nodal displacements. From Eq. (24), the incremental form of the strain–displacement relationship is
\[
\delta E = (B_L + B_{NL}(u))\delta u. \tag{25}
\]

From the principle of virtual displacement, the virtual work $\delta U$ is
\[
\delta U = \delta u^T M \ddot{u} + \delta u^T A \ddot{u} + \int_{V_0} \delta E^T \dot{S} dV - \dot{\delta u}^T F_{\text{ext}} - \delta u^T R = 0, \tag{26}
\]

where $V_0$ is the domain of the initial configuration, $F_{\text{ext}}$ the vector of external loads, $R$ the contact reaction vector, $M$ the mass matrix, $A$ the damping matrix, $u$ the velocity vector and $\ddot{u}$ the acceleration vector. In view of Eqs. (21) and (25), it comes
\[
\delta S = D\delta E = D(B_L + B_{NL}(u))\delta u, \tag{27}
\]
where $D$ denotes the usual material secant tangent matrix, deduced from the stress–strain tangent operator $D$ (Eq. (21)) due...
The vector of internal forces is defined by

\[ \mathbf{D} = \begin{bmatrix} D_{1111} & D_{1112} & D_{1113} & D_{1112} \\ D_{1122} & D_{1123} & D_{1221} & D_{1223} \\ D_{1133} & D_{1321} & D_{1323} & D_{1331} \\ D_{2212} & D_{2213} & D_{2312} & D_{2313} \\ D_{2222} & D_{2233} & D_{2321} & D_{2331} \\ D_{2233} & D_{2323} & D_{2333} & D_{3331} \end{bmatrix} \] . (28)

Substituting \( \delta E \) from Eq. (25) into Eq. (26) results in

\[ \delta U = \delta \mathbf{u}^T \mathbf{M} \delta \mathbf{u} + \delta \mathbf{u}^T \mathbf{A} \delta \mathbf{u} + \delta \mathbf{u}^T \int_{V_0} (\mathbf{B}_L + \mathbf{B}_{NL}(\mathbf{u}))^T \mathbf{S} \, dV \]

\[ - \delta \mathbf{u}^T \mathbf{F}_{\text{int}} - \delta \mathbf{u}^T \mathbf{R} = 0. \] (29)

The vector of internal forces is defined by

\[ \mathbf{F}_{\text{int}} = \int_{V_0} (\mathbf{B}_L + \mathbf{B}_{NL}(\mathbf{u}))^T \mathbf{S} \, dV . \] (30)

Since \( \delta \mathbf{u} \) is arbitrary, the following set of nonlinear equations is obtained:

\[ \mathbf{M} \ddot{\mathbf{u}} + \mathbf{A} \dot{\mathbf{u}} + \mathbf{F}_{\text{int}} - \mathbf{F}_{\text{ext}} - \mathbf{R} = 0. \] (31)

It is remarked that the stiffness effect is taken into account through the internal forces vector \( \mathbf{F}_{\text{int}} \). Eq. (31) can be transformed to

\[ \mathbf{M} \ddot{\mathbf{u}} + \mathbf{A} \dot{\mathbf{u}} + \mathbf{F}_{\text{int}} - \mathbf{F}_{\text{ext}} - \mathbf{A} \dot{\mathbf{u}} = 0. \] (32)

to be solved with the initial conditions at \( t = 0 \),

\[ \mathbf{u} = \mathbf{u}_0 \quad \text{and} \quad \dot{\mathbf{u}} = 0. \] (33)

Taking the derivative of \( \mathbf{F}_{\text{int}} \) with respect to the nodal displacements \( \mathbf{u} \) yields the tangent stiffness matrix:

\[ \mathbf{K} = \frac{\partial \mathbf{F}_{\text{int}}}{\partial \mathbf{u}} = \mathbf{K}_e + \mathbf{K}_a + \mathbf{K}_d, \] (34)

where \( \mathbf{K}_e \), \( \mathbf{K}_a \) and \( \mathbf{K}_d \) stand, respectively, for the elastic stiffness matrix, the geometric stiffness (or initial stress stiffness) matrix and the initial displacement stiffness matrix:

\[ \mathbf{K}_e = \int_{V_0} \mathbf{B}_L^T \mathbf{D} \mathbf{B}_L \, dV , \] (35)

\[ \mathbf{K}_a = \int_{V_0} \frac{\partial \mathbf{B}_{NL}^T}{\partial \mathbf{u}} \mathbf{S} \, dV , \] (36)

\[ \mathbf{K}_d = \int_{V_0} (\mathbf{B}_{NL}^T \mathbf{D} \mathbf{B}_{NL} + \mathbf{B}_{NL}^T \mathbf{D} \mathbf{B}_L + \mathbf{B}_{NL}^T \mathbf{D} \mathbf{B}_{NL} \mathbf{D} \mathbf{B}_{NL}) \, dV . \] (37)

3.2. First order time integration

Eq. (32) has to be integrated between consecutive time configurations at time \( t \) and \( t + \Delta t \). Usually a Newmark scheme based on a second order approximation is used. However, in impact problems, this scheme may lead to no physical energy blow-up as shown in [4]. At the moment of a sudden change of contact conditions (impact, release of contact), the velocity and acceleration are not continuous in time, and excessive regularity of constraints may lead to errors. For this reason, Jean [8] proposed a first order time scheme which is used in this work. This scheme is based on the following approximations:

\[ \int_t^{t+\Delta t} \mathbf{M} \dot{\mathbf{u}} = \mathbf{M} (\mathbf{u}^{i+\Delta t} - \mathbf{u}^i) , \] (38)

\[ \int_{t}^{t+\Delta t} \mathbf{F} \, dt = \Delta t ((1 - \zeta) \mathbf{F}^i + \zeta \mathbf{F}^{i+\Delta t}) , \] (39)

\[ \int_{t}^{t+\Delta t} \mathbf{R} \, dt = \Delta t \mathbf{R}^{i+\Delta t} , \] (40)

\[ \mathbf{u}^{i+\Delta t} - \mathbf{u}^i = \Delta t ((1 - \theta) \dot{\mathbf{u}}^i + \theta \dot{\mathbf{u}}^{i+\Delta t}) , \] (41)

with the parameters \( 0 < \zeta, \theta \leq 1 \). In the iterative solution procedure, all the values at time \( t + \Delta t \) are replaced by the values of the current iteration \( i+1 \); for example, \( \mathbf{F}^{i+\Delta t} = \mathbf{F}^{i+1} \).

A standard approximation of \( \mathbf{F}^{i+1} \) is

\[ \mathbf{F}^{i+1} = \mathbf{F}_{\text{int}}^{i+1} + \frac{\partial \mathbf{F}}{\partial \mathbf{u}} (\mathbf{u}^{i+1} - \mathbf{u}^i) + \frac{\partial \mathbf{F}}{\partial \mathbf{u}} (\dot{\mathbf{u}}^{i+1} - \dot{\mathbf{u}}^i) \]

\[ \mathbf{F}_{\text{int}}^{i+1} - \mathbf{K}^i \mathbf{A} \mathbf{u} = \mathbf{A}^i \mathbf{u} . \] (42)

Finally, we obtain the recursive form of Eq. (32) in terms of displacements:

\[ \tilde{\mathbf{K}}^i \mathbf{u} = \tilde{\mathbf{F}}^i + \tilde{\mathbf{F}}_{\text{acc}}^{i+1} + \mathbf{R}^{i+1} , \]

\[ \mathbf{u}^{i+1} = \mathbf{u}^i + \mathbf{A} \mathbf{u} , \] (43)

where the so-called effective terms are given by

\[ \tilde{\mathbf{K}}^i = \mathbf{K}^i + \frac{\zeta}{\partial \Delta t^2} \mathbf{M}^i , \] (44)

\[ \tilde{\mathbf{F}}_{\text{acc}}^{i+1} = - \frac{1}{\partial \Delta t} \mathbf{M}^i (\mathbf{u}^{i+1} - \mathbf{u}^i - \Delta \dot{\mathbf{u}}^i) , \] (45)

\[ \tilde{\mathbf{F}}^i = (1 - \zeta) (\mathbf{F}_{\text{int}}^i + \mathbf{F}_{\text{ext}}^i) + \zeta (\mathbf{F}_{\text{int}}^i + \mathbf{F}_{\text{ext}}^{i+\Delta t}) . \] (46)

To complete the time step, the velocity is updated according to

\[ \dot{\mathbf{u}}^{i+\Delta t} = \left( 1 - \frac{1}{\theta} \right) \dot{\mathbf{u}}^i + \frac{1}{\partial \Delta t} (\mathbf{u}^{i+\Delta t} - \mathbf{u}^i) . \] (47)

Eq. (43) is strongly nonlinear, because of large rotations and large deformations involved. Besides, in multibody contact/impact problems, unilateral contact and friction, characterized by inequalities, are non-smooth phenomena. To solve this
equation instead of considering all nonlinearities at the same time, a strategy was proposed in [18] which consists in separating the nonlinearities so as to overcome the complexity of calculation and to improve the numerical stability. As \( \Delta u \) and \( \mathbf{R} \) are both unknown, Eq. (43) cannot be directly solved. First, the vector \( \mathbf{R} \) is determined using the bi-potential method in a reduced system, involving only contact nodes. The reader can refer to [5,6,18,19] for more details on the bi-potential method. Then, the vector \( \Delta u \) is computed over the whole structure, using contact reactions as external loadings. It is important to note that, as opposed to the penalty method or Lagrange multiplier method, the bi-potential method neither changes the global stiffness matrix nor increases the number of degrees of freedom. This interesting feature makes it easy to implement contact and friction problems in existing general-purpose finite element codes. In addition, the solution procedure is more stable because of the separation of nonlinearities and improved numerical algorithms for calculation of contact reactions.

3.3. Energy computation

When the displacement and the velocity fields are determined, the energies can be computed. From Eq. (14), we calculate the strain energy density \( W \) at the elemental level through Gauss integration formula. The total elastic strain energy of the bodies (discretized by \( n_{el} \) finite elements) is

\[
E_e = \sum_{e=1}^{n_{el}} \int_{V_e} W \, dV. \tag{48}
\]

The total kinetic energy can be calculated at the global level by

\[
E_k = \frac{1}{2} \mathbf{u}^T \mathbf{M} \mathbf{u}. \tag{49}
\]

Finally, the total energy of the system of solids is the sum of elastic and kinetic energies:

\[
E_t = E_e + E_k. \tag{50}
\]

4. Numerical results

The algorithms presented above have been implemented in the finite element code FER/Impact [20]. Many application examples, in static or quasi-static cases, have been carried out using the present method [6,18].

To illustrate the results of the contact/impact simulation using the algorithm described above, we consider here one example application, initially proposed by Love and Laursen [21] on considering linear elastic material model. We assume that no damping exists except for Coulomb friction between contact surfaces, i.e., \( \mathbf{A} = 0 \) in Eq. (44).

The problem consists of two three-dimensional hyperelastic blocks (Fig. 1) that impact with relative tangential motion. Normalized units are used in this example. The base of the larger block is fixed, and the smaller block has an initial rigid-body velocity of \([0.0, 1.5, -1.0]\) that initiates a glancing impact. The larger block initially occupies the cubic space defined by diagonal corner points \([0, 0, 0]\) and \([2.0, 2.0, 1.0]\) and the smaller block is similarly defined by points \([0.5, 0.0, 1.25]\) and \([1.5, 1.0, 2.25]\). The density is 100. The initial shear modulus \( G \) and bulk modulus \( K \) are chosen as in [21]: \( G = 5000, K = 3333 \). Other parameters of the 3\( \varepsilon \) model are: \( a_1 = 50, a_2 = 50, a_3 = 0, a_4 = 2000 \) and \( a_5 = 100 \). The total simulation time is \( 1 \) scaled time unit and the numerical parameters are: \( \Delta t = 10^{-2}, \xi = 0.5 \). To investigate the frictional effects on the energy dissipation, different coefficients of Coulomb friction are used: \( \mu = 0.0 \) (frictionless), \( 0.2, 0.5, 0.8 \).

Figs. 2 and 3 show the deformed shapes at time \( t = 0.4 \) and 0.6 with the friction coefficient \( \mu = 0.2 \). The isocontours represent...
the distribution of the von-Mises stress inside the blocks (the maximum value is 1330 unit of stress). Similarly, Figs. 4 and 5 show the case with \( \mu = 0.8 \). The maximum value of the von-Mises stress is 3330 unit of stress. These plots highlight the impact of the friction coefficient on the stress level and relative slips. The case with \( \mu = 0.2 \) corresponds to a sliding contact status while the case with \( \mu = 0.8 \) corresponds to almost a sticking contact status. This result is in good agreement with the observation reported by Love and Laursen [21].

Figs. 6–8 show the plots of the kinetic energy \( E_k \), the elastic strain energy \( E_e \) and the total energy \( E_t \) as a function of time. We observe that the total energy is quite well conserved in the case of frictionless contact (Fig. 8). However, in the case of frictional contact, the total energy decreases. So the total energy is dissipated by frictional effects as expected. It is worth mentioning that the dissipated energy is quantified. It is also interesting to examine another question: is the dissipated energy proportional to the friction coefficient? The answer is negative according to numerical results. The proof is illustrated in Fig. 8 where we observe almost the same dissipated energy even for two different friction coefficients (\( \mu = 0.2, 0.5 \)). In addition, the dissipated energy is less for \( \mu = 0.8 \) than for \( \mu = 0.2 \) or 0.5. In fact, when the friction coefficient increases, the friction forces increase too. However, the tangential slips decrease. We know that the dissipated energy depends not only on the friction forces but also on the tangential slips on the contact surface.

Figs. 9 and 10 show, respectively, the evolution of the von Mises stress at point A and \( \sigma_{zz} \) at point C (see Fig. 1). It can
be seen that when the friction coefficient increases, the stress level is bigger. The trajectory of point B in the plane BCD (see Fig. 1) is depicted in Fig. 11. We observe that the amplitude of the displacements increases with the friction coefficient as expected.

5. Conclusion

In this paper, we have proposed a finite element implementation of the $3\varepsilon$ model and a numerical application to the case of dynamic contact with Coulomb friction between two hyperelastic blocks. From numerical experiments, we have found that:

- The total energy is well conserved for frictionless contact.
- The algorithm allows for the quantification of the physical energy dissipation by friction.
- The dissipated energy is not a monotonic function of the friction coefficient.
- The tumbling behavior is numerically recovered.
References