A closed form solution for the uniaxial tension test of biological soft tissues

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1. Introduction

To investigate internal deformation and stress of biological soft tissues such as ligaments, tendons or arterial walls, anisotropic hyperelastic constitutive laws are often used in the framework of finite element analysis \cite{1,2,3}. The most used strain-energy functions take a power law form \cite{4} or present an exponential behavior \cite{5,6}. More recently, Balzani et al. \cite{7} have proposed polyconvex strain energy functions combining an exponential form with a power law to take care of the tissues behavior in the low load domain. More realistic models have been also recently developed to capture the inter-fiber angle change by adding to the strain energy the contribution of the fiber-matrix shear interaction \cite{8}. In general, the anisotropy can be represented via the introduction of a so-called structural tensor, which allows a coordinate-invariant formulation on the constitutive equations \cite{9,10,11,12}. It is usually assumed that anisotropy is due to the collagen fibers behavior \cite{12}, while the ground substance, or matrix, behaves in an isotropic manner, so the energy densities modeling transversely isotropic and orthotropic soft tissues are separated into isotropic and anisotropic parts \cite{1,7}. Each anisotropic density refers to a preferred direction of the material. For example, in Holzapfel–Gasser–Ogden’s constitutive law \cite{6,13}, two transversely isotropic energies with two distinct preferred directions corresponding to two superposed collagen fiber families.

This paper concerns the modeling of biological soft tissues in the framework of anisotropic hyperelasticity. A closed form solution is proposed in the special case where uniaxial unconstrained tension loading is applied to a strip modeled by Holzapfel–Gasser–Ogden’s (HGO) strain energy. The classical Cardano’s formula is used to calculate the solution. This solution is investigated for different values of $\beta$ which represents the angle between the collagen fibers and the circumferential direction. This study allows for the understanding of the reason why a one-to-one correspondence does not exist between the principal stretch and the fourth invariant of the right Cauchy–Green deformation tensor. It is demonstrated that the relationship becomes non-bijective when $\beta$ is greater than a critical angle of $54.73^\circ$. The HGO model is implemented into an in-house finite element program and numerical results are in good agreement with theoretical ones.

2. HGO hyperelastic model

Most energy densities used to model transversely isotropic and orthotropic soft tissues take a power law form \cite{4} or present an exponential behavior \cite{5,6}. It is usually assumed that anisotropy is due to the collagen fibers behavior \cite{12} while the ground substance, or matrix, behaves in an isotropic manner. The energy densities modeling transversely isotropic and orthotropic soft tissues are thus separated into isotropic and anisotropic parts \cite{1}

$$W = W_{iso} + \sum_{n=1}^{N} W_{ani}$$

Each anisotropic density $W_{ani}$ refers to a preferred direction of the material. The number of fiber families $n$ is generally set to 1 to model tissues as ligaments or tendons while it is set to 2 to represent the behavior of arterial walls. For example, to model the embedded collagen fibers of soft biological arterial tissues,
Holzapfel–Gasser–Ogden’s constitutive law [6,12] superposes two transversely isotropic energies with two distinct preferred directions $a^1$ and $a^2$ corresponding to two fiber families:

$$a^1 = \left\{ \begin{array}{ccc} c & 0 & 0 \\ s & 1 & 0 \\ 0 & 0 & 1 \end{array} \right\}, \quad a^2 = \left\{ \begin{array}{ccc} c & 0 & 0 \\ -s & 1 & 0 \\ 0 & 0 & 1 \end{array} \right\} \quad \text{with } c = \cos(\beta), \quad s = \sin(\beta)$$

(2)

The phenomenological angle $\beta$ represents the angle between the collagen fibers and the circumferential direction for a strip extracted, for example, from the media of a human abdominal aorta (Fig. 1). $\beta$ is predicted to be 43° 39’ in the work of Balzani et al. [7]. In our work, we consider $\beta$ as a model parameter varying from 0° to 90°.

The strain measure adopted here is the Green–Lagrangian strain tensor

$$E = \frac{1}{2} (C - I)$$

(3)

where $I$ is the second order unit tensor, $C = F^T F$ is the right Cauchy–Green deformation tensor and $F$ the transformation gradient defined by

$$F = \frac{\partial x}{\partial \xi} = I + \frac{\partial u}{\partial \xi}, \quad J = \det(F) > 0$$

(4)

$X, \xi$ and $u$ represent respectively the reference and the current positions and the displacement vector of a material point. According to Zhang–Rychlewski’s theorem [15], the condition of material symmetry is satisfied if structural tensors are additionally included in the strain energy density representation. Transversely isotropic densities can then be expressed with the three invariants $I_1, I_2$ and $I_3$ of the right Cauchy–Green deformation tensor and two additional mixed invariants $J_4$ and $J_5$ [9–11]

$$I_1 = \text{tr}(C), \quad I_2 = \text{tr}([\text{cof}(C)], \quad I_3 = \text{det}(C), \quad J_4 = \text{tr}(CM), \quad J_5 = \text{tr}(C^2M)$$

(5)

where $\text{cof}(C)$ denotes the co-factor matrix of $C$ and $M$ is the so-called structural tensor representing the transverse-isotropy group and referring to a preferred direction $a$ of the material $M = a \otimes a$

(6)

It is noted that (5) and (6) give

$$J_4 = \text{tr}(F^T F a \otimes a) = |Fa|^2$$

(7)

The double brackets represent the usual Euclidian norm. The square root of $J_4$ represents thus the stretch in the fiber direction. It can also be interpreted as the radial coordinate of $Fa$ in a cylindrical coordinate system where the polar angle $\theta$ represents the deformed angle between the collagen fibers and the circumferential direction (Fig. 2).

In what follows, $J_4$ and $\theta$ will be used to solve the equilibrium equations in an analytical manner. In the case of hyperelastic materials, there exists an elastic potential function $W$ which is a scalar function of the strain tensors. Generally, soft biological tissues are assumed to be incompressible. The incompressibility is characterized in terms of principal stretches $\lambda_1, \lambda_2$ and $\lambda_3$ by $J = \det(F) = \lambda_1 \lambda_2 \lambda_3 = 1$ related to an extra pressure $p$. The second Piola–Kirchhoff stress tensor $S$ and the corresponding Cauchy stress tensor $\sigma$ are given by

$$S = 2 \frac{\partial W}{\partial C} \cdot p C^{-1}, \quad \sigma = J^{-1} F S F^T$$

(8)

To uncouple the deviatoric part to the dilatational part of the response, the volume preserving part $F = J^{1/3} F$ of the deformation is introduced [1]. The modified invariants related to $C = F^T F = J^{-2/3} C$ are expressed from (5) by

$$I_1 = I_1 J^{-2/3}, \quad I_2 = I_2 J^{-4/3}, \quad \lambda_4^2 = J_4 J^{2/3}, \quad \lambda_5^2 = J_5 J^{4/3}$$

(9)

The exponential type HGO density adopted in this work uses these modified invariants as follows:

$$W_{\text{uni}} = \begin{cases} k_1 \frac{\lambda_4^2}{2k_2} \left( \frac{\lambda_4^2}{\lambda_4^2 - 1} \right)^p - 1 \quad & \text{if } \lambda_4^2 \geq 1 \\ 0 \quad & \text{if } \lambda_4^2 < 1 \end{cases}$$

(10)

This energy density is case sensitive with respect to $\lambda_4^2$ because the case of $\lambda_4^2 < 1$ represents the shortening of the fibers which is assumed to generate no stress. The proof of convexity of (10) with respect to $F$ is given in [4,7]. The non-collagenous matrix of the media is modeled by the neo-Hookean isotropic density

$$W_{\text{neo}} = c_1 (I_1 - 3)$$

(11)

It is noted that the volumetric-isochoric split of the above HGO model does only hold for (quasi) incompressible deformations. An extension to compressible deformations would require that the volumetric part of the strain energy function includes a dependency on the structural tensor. This is proved recently by Guo et al. [16] where a simple compressible anisotropic analytical model is developed.

In order to conduct analytical solutions, we consider a uniaxial unconstrained tension test as depicted in Fig. 3.
This loading is usually applied to biological soft tissues for identifying parameters of the material model. In this test, the parameters are set to \( c_1 = 10.2069 \text{kPa} \), \( k_1 = 0.0017 \text{kPa} \) and \( k_2 = 882.847 \). These values are proposed in [7] to fit numerical models with experimental data. As the test leads to homogeneous deformations, the equilibrium equations are automatically satisfied and the boundary conditions related to the free traction surfaces are

\[
\sigma_{22} = \sigma_{33} = 0 \quad (12)
\]

The principal stretches \( \lambda_1, \lambda_2 \) and \( \lambda_3 \) will be calculated in next sections by using an analytical approach and compared to finite element results. The incompressibility constraint is included in the finite element model by using a penalty term of the form \( k(J - 1)^2/2 \) with \( k = 10^5 \).

3. Theoretical study

By considering (1), (8)–(11), it is easily demonstrated that

\[
J_4 \geq 1: \sigma = -p \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} + 2c_1 \begin{pmatrix} \lambda_1^2 & \lambda_1^2 \lambda_2^2 \\ \lambda_1^2 \lambda_2^2 & \lambda_2^4 \end{pmatrix} \begin{pmatrix} \frac{1}{3} & 1 \\ 1 & 1 \end{pmatrix}
+ 4k_1(j_4 - 1) \begin{pmatrix} \lambda_1^2 \lambda_2^2 & 0 \\ 0 & \lambda_1^2 \lambda_2^2 \end{pmatrix} \begin{pmatrix} \frac{1}{3} & 1 \\ 1 & 1 \end{pmatrix}
\] (13)

\[
J_4 < 1: \sigma = -p \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} + 2c_1 \begin{pmatrix} \lambda_1^2 & \lambda_1^2 \lambda_2^2 \\ \lambda_1^2 \lambda_2^2 & \lambda_2^4 \end{pmatrix} \begin{pmatrix} \frac{1}{3} & 1 \\ 1 & 1 \end{pmatrix}
\] (14)

By considering \( \sigma_{33} = 0 \) to express the pressure \( p \) in terms of the principal stretch and taking into account the incompressibility condition \( J = 1 \), we obtain

\[
J_4 \geq 1: \begin{cases}
\sigma_{11} = 2c_1(\lambda_1^2 - 1) + 2k_1(\lambda_1^2 \lambda_2^2 - 1) + 2k_1 \lambda_1^4 \lambda_2^6 (j_4 - 1)^2 \\
\sigma_{22} = 2c_1(\lambda_1^2 - 1) + 2k_1(\lambda_1^2 \lambda_2^2 - 1) + 2k_1 \lambda_1^4 \lambda_2^6 (j_4 - 1)^2 \\
\sigma_{33} = 0
\end{cases}
\] (15)

\[
J_4 < 1: \sigma_{11} = 2c_1(\lambda_1^2 - 1) + 2k_1(\lambda_1^2 \lambda_2^2 - 1), \quad \sigma_{22} = 2c_1(\lambda_1^2 - 1) + 2k_1(\lambda_1^2 \lambda_2^2 - 1) = 0
\] (16)

To determine the principal stretches in an analytical manner, we will demonstrate that the equilibrium equations can be transformed into a third degree polynomial form. The polynomial equation can then be algebraically solved by using Cardano’s formula. As the stresses expressions (15) and (16) depend on the value of \( J \), we logically start by considering two different cases.

3.1. Case \( J_4 \geq 1 \)

The free traction surfaces condition \( \sigma_{22} = 0 \) applied to (15) gives

\[
c_1(\lambda_2^2 - 1) + 2k_1\lambda_1^2 \lambda_2^2 (j_4 - 1)^2 = 0
\] (17)

Besides, it results from (2) and (5) that

\[
(\lambda_1^4 d_4^{-1/2} + \lambda_2^4 d_4^{-1/2})^2 = 1
\] (18)

This suggests that a cylindrical coordinate system could be appropriate to solve the non-linear equation (17). This cylindrical coordinate system is defined by setting \( d_4 \) as the radial coordinate and by introducing the polar angle \( \theta \) representing the deformed angle between the collagen fibers and the circumferential direction (Fig. 2)

\[
\cos(\theta) = \lambda_1 d_4^{-1/2}, \quad \sin(\theta) = \lambda_2 d_4^{-1/2}
\] (19)

If (19) is reported in (17), the principal stretches \( \lambda_1 \) and \( \lambda_2 \) are replaced by two new variables \( \theta \) and \( f_4 \). Eq. (17) becomes then a cubic polynomial equation:

\[
\cos(\theta) - \cos^3(\theta) = s^2 f_4^{3/2} = A
\] (20)

To calculate the roots of (20) with Cardano’s formula, three different cases have to be discussed with respect to the sign of the discriminant \( A \)

\[
A = A^2 / 4 - 1 / 27
\] (21)

Case a:

\[
A > 0 \quad \Rightarrow \quad A < \frac{2}{3 \sqrt{3}}
\] (22)

It results from the definition (20) of \( A \)

\[
A \leq s^2 c
\] (23)

We consider the function of \( \theta \) representing the left-hand side of (20)

\[
f(\theta) = \cos(\theta) - \cos^3(\theta) = \cos(\theta) \sin^2(\theta), \quad \theta \in [0, 90] \]
(24)

It is easy to demonstrate that the maximum value of \( f \) is equal to \( 2/\sqrt{3} \) and is reached if \( \theta = \beta_e \) with (Fig. 4)

\[
\beta_e = \arccos(\sqrt{3}/3) \approx 54.73^\circ
\] (25)

It then results from (23) that

\[
A \leq 2/3 \sqrt{3}
\] (26)

This result is contradictory with (22). Consequently, Case a will never occur.

Case b:

\[
A = 0 \quad \Rightarrow \quad A = \frac{2}{3 \sqrt{3}}
\] (27)

In this case, Cardano’s formula gives one single and one double real root

\[
\cos(\theta_1) = -2/\sqrt{3}, \quad \cos(\theta_2) = \sqrt{3}/3
\] (28)

According to (24), the roots must be positive. The single root \( \cos(\theta_1) \) is thus not acceptable. However, the double root \( \cos(\theta_2) \) is acceptable as it is included in the range \([0, 1]\). The corresponding

![Fig. 4. Left-hand side of (20).](image-url)
stretches are deduced from (19)
\[ \lambda_1 = J_1^{1/2}/(\sqrt{3}c), \quad \lambda_2 = 2J_4^{1/2}/(\sqrt{3}s) \]  
(29)

As the maximum value of \( f \) is reached at \( \theta = \beta_i \) (Fig. 4), (27) gives
\[ A = f(\beta_i) = f(\beta) \]  
(30)

Besides, it results from the definitions (20) and (24) of \( A \) and \( f \) that
\[ \text{If } J_4 > 1: A < f(\beta) \quad \text{If } J_4 = 1: A = f(\beta) \]  
(31)

The conditions (30) and (31) are together consistent if and only if \( \beta = \beta_i \) and \( J_4 = 1 \). It is then deduced from (29) that Case b gives thus a double root corresponding to the evident solution
\[ \lambda_1 = \lambda_2 = 1. \]

Case c:
\[ A < 0 \quad \Rightarrow \quad A < \frac{2}{\sqrt{3}} \]  
(32)

In this case, according to Cardano’s formula, there exist three distinct real roots
\[ \cos(\theta_i) = \frac{2\sqrt{3}}{3} \cos \left( \frac{z + 2i(1-1)i\pi}{3} \right), \quad z = \arccos \left[ \frac{3\sqrt{3}\lambda}{2} \right], \quad i = 1, 2, 3 \]  
(33)

The locations on the trigonometric circle of the three angles used in (33) are depicted in Fig. 5. To determine the acceptable roots, we first notice that (20) implies that \( A \) is positive. It then results from (32) and (33) that
\[ 0 < \cos(\theta_1) < \frac{\sqrt{3}}{3} < \cos(\theta_1) < 1, \quad \cos(\theta_2) < 0 \]  
(34)

It is noted that the second root \( \cos(\theta_2) \) is negative and must be excluded. Among the two positive roots \( \cos(\theta_1) \) and \( \cos(\theta_2) \), we propose to demonstrate that the third is unacceptable except if \( J_4 \) is equal to 1. To prove it, we remind that a tensile loading test is considered. It then results from (19) and (33) that
\[ \lambda_1 = 2J_4^{1/2} \cos \left( \frac{3\pi}{3} \right), \quad (\sqrt{3}) \geq 1 \]  
(35)

After some algebra, it can be demonstrated that (35) leads to \( J_4 \leq 1 \). As the case of \( J_4 \geq 1 \) is considered, we have \( J_4 = 1 \). The root \( \cos(\theta_2) \) is thus excluded except if \( J_4 = 1 \). In conclusion, if \( \lambda \) is negative and \( J_4 = 1 \), there exists only one single root defined by \( \cos(\theta_1) \). If \( \lambda \) is negative and \( J_4 = 1 \), there exists two distinct single roots defined by \( \cos(\theta_1) \) and \( \cos(\theta_2) \).

3.2. Case \( J_4 < 1 \)

This case is easier to study than the previous one because the orthotropic behavior is not taken into account and the model is reduced to the neo-Hookean energy density according to Eq. (10). By using the stress expression (16) in conjunction with the free traction surface loading \( \sigma_{zz} = 0 \), the neo-Hookean solution is directly obtained
\[ \lambda_2 = 1/\sqrt{\lambda_1} \]  
(36)

To study the dependence of \( \lambda_1 \) and \( \lambda_2 \) on \( J_4 \) in the same way as for the case \( J_4 \geq 1 \), it is needed to solve again a cubic polynomial equation. This equation is deduced from (18) and (36)
\[ g(\lambda_1) = \lambda_1^3 - J_4 c^2 \lambda_1 + t^2 = 0 \quad \text{with} \quad t = s/c \]  
(37)

This function admits a local maximum and a local minimum as shown in Fig. 6. The local maximum and minimum points are defined respectively by \( \left( -\sqrt{J_4/3c^2}, t^2 + \sqrt{4J_4^2/27c^6} \right) \) and \( \left( \sqrt{J_4/3c^2}, t^2 - \sqrt{4J_4^2/27c^6} \right) \).

To determine the real roots of \( g \), Eq. (37) is solved analytically by using again Cardano’s formula. Three different cases occur depending on the sign of the discriminant \( \Delta_t \)
\[ \Delta_t = t^4 - 4J_4^2/27c^6 \]  
(38)

Case a:
\[ \Delta_t > 0 \quad \Rightarrow \quad t^2 > \sqrt{4J_4^2/27c^6} \]  
(39)

In this case, Cardano’s formula indicates that one root is a real number while the two others are conjugate complex numbers. Besides, it is reminded that the condition \( \lambda_1 \geq 1 \) must be satisfied because a tensile load is considered. As \( g \) increases from \( -\infty \) to \( t^2 + \sqrt{4J_4^2/27c^6} > 0 \) for \( \lambda_1 \) varying from \( -\infty \) to \( -\sqrt{J_4/3c^2} \), the real root is negative and should not be selected. Finally, using definitions (24) and (25), the condition (39) can be simplified to
\[ J_4 < [f(\beta)/f(\beta_i)]^{2/3} \]  
(40)

Case b:
\[ \Delta_t = 0 \quad \Rightarrow \quad t^2 = \sqrt{4J_4^2/(27c^6)} \]  
(41)

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**Fig. 5.** Location of \( \pi/3 \), \( (\pi + 2\pi)/3 \) and \( (\pi + 4\pi)/3 \) on the trigonometric circle.

**Fig. 6.** Graph of \( g(\beta = 50, J_4 = 0.5) \).
In this case, Cardano's formula gives a single and a double real root:

Single root : \( \lambda_1 = -3s^2/J_4 \), Double root : \( \lambda_1 = 3s^2/2J_4 \) \( (42) \)

The single root is negative and has to be excluded. By using (41), it is noted that the double root corresponds to the minimum of the \( g \) function. To make this root acceptable, the condition \( \lambda_1 \geq 1 \) must hold such that:

\[ (t^4/A)^{1/3} \geq 1 \iff t \geq \sqrt[3]{2} \iff \beta \geq \beta_k = \arccos(\sqrt{3}/3) \approx 54.73^\circ \] \( (43) \)

The double root defined by (42) is thus solution of Eq. (37) if and only if the phenomenological angle \( \beta \) is greater than the critical angle \( \beta_k \) defined by (43). It is worth noting that this critical angle has also been found by Guo et al. with a composites-based hyperelastic constitutive model \( (17) \). Moreover, it results from (19), (36), (41), (42) that:

\[ \cos \theta = \cos \beta_c, \quad \sin \theta = \sin \beta_c \Rightarrow \theta = \beta_c \] \( (44) \)

It is thus remarkable that the deformed angle \( \theta \) is equal to the critical angle \( \beta_c \) in the case of a double root occurrence corresponding to an initial angle \( \beta \) greater than \( \beta_c \). Finally, it is noted that condition (41) can be simplified in the same way as for Case a:

\[ J_4 = (f/\beta/c)3/3 \] \( (45) \)

Case c:

\[ A_1 < 0 \iff t^2 < (4J_4^2/(27c^2)) \] \( (46) \)

In this case, Cardano's formula provides three distinct real roots expressed trigonometrically:

\[ \lambda_{1b} = 2\sqrt{J_4/3c^2} \sin(\gamma/3), \quad \lambda_{1c} = 2\sqrt{J_4/3c^2} \cos(\gamma/3 + 2\pi/3) \]
\[ \lambda_{1c} = 2\sqrt{J_4/3c^2} \cos(\gamma/3) \] \( (47) \)

\[ \gamma = \arccos(-t^2/27c^2/(4J_4^2)) \] \( (48) \)

To determine the acceptable roots, we first deduce from (46)–(48) that:

\[ 0 < \lambda_{1c} < J_4/(3c^2) < \lambda_{1b} < J_4/c^2, \quad -2J_4/(3c^2) < \lambda_{1b} < -J_4/c^2 \] \( (49) \)

The second root \( \lambda_{1b} \) is negative and cannot be selected. To determine whether or not the positive roots \( \lambda_{1a} \) and \( \lambda_{1b} \) are greater than one, according to the condition of a tension test, two cases are considered:

Case c1:

\[ \beta < \beta_k = \arccos(\sqrt{3}/3) \] \( (50) \)

Because the cosine function decreases in the interval \([0, \pi/2] \), (50) yields to:

\[ 3c^2 > 1 \Rightarrow 0 < J_4/(3c^2) < 1 \] \( (51) \)

Eq. (51) means that the minimum of \( g \) is reached between 0 and 1. We then apply the intermediate value theorem to \( g \) and we account for condition (46):

\[ g(J_4/3c^2) = t^2 - (4J_4^2/27c^2) < 0, \quad g(1) = 1 - J_4/c^2 > 0 \]

\[ \Rightarrow \lambda_{1a} \in \left( J_4/3c^2, 1 \right) \quad g(\lambda_{1a}) = 0 \] \( (52) \)

In the same way, it is concluded that \( \lambda_{1c} < 1 \). Consequently, the two positive roots are unacceptable.

Case c2:

\[ \beta \geq \beta_k = \arccos(\sqrt{3}/3) \] \( (53) \)

As the tangent function increases in the interval \([0, \pi/2] \), (46) and (53) yield to:

\[ t^2 \geq 2 \Rightarrow \sqrt{J_4/(3c^2)} > 1 \] \( (54) \)

To conclude that \( \lambda_{1c} > 1 \), we consider Eq. (54) and we apply again the intermediate value theorem in the same way as in Case c1. It results next from (49) that \( \lambda_{1a} > 1 \). Consequently, the two positive roots \( \lambda_{1a} \) and \( \lambda_{1c} \) defined by (47) are acceptable. Finally, by reasoning in the same manner as for Case a, the condition (46) can be simplified to:

\[ J_4 > (y/f(\beta/c))^2/3 \] \( (55) \)

4. Analytical and numerical results

The analytical formula applied to the uniaxial tension test determines the evolution of \( \lambda_2 \) and \( \theta vs. J_4 \) as shown in Fig. 7. If \( \beta < \beta_c \) \( (e.g., \beta = 20^\circ) \), there is no solution corresponding to \( J_4 < 1 \). The biological tissue can thus not be shortened in the fiber direction. On the contrary, if \( \beta > \beta_c \) \( (e.g., \beta = 70^\circ) \), it is observed that the biological tissue can be shortened in the fiber direction. However, the correspondence between \( \lambda_2 \) and \( J_4 \) is not one-to-one because of the existence of two distinct solutions. These two solutions are given by (47) and (48). It is finally noted that the curve related to \( \beta = 70^\circ \) starts with \( J_4 \) equal to a minimum value defined by (45). This minimum value corresponds to the double root defined by (42). Eq. (45) shows that the minimum value of \( J_4 \) only depends on the initial angle \( \beta \). As the function \( f \) decreases if \( \beta > \beta_c \) (Fig. 4), this minimum value decreases if \( \beta \) increases. This trend is well observed in Fig. 7 (left). The minimum values of \( J_4 \) are respectively equal to 0.98, 0.85 and 0.58 for \( \beta = 60^\circ \) and 80 \( ^\circ \). The occurrence of a double root with \( J_4 < 1 \) corresponds to a remarkable result given by (44), i.e., the deformed angle \( \theta \) is equal to the critical angle \( \beta_c \), when the minimum value of \( J_4 \) given by (45) is reached as shown in Fig. 7 (right). Fig. 8 shows the variation of the cylindrical coordinates \( (J_4, \theta) \) vs. \( \lambda_1 \) for two values of \( \beta \) \( (20^\circ \) and \( 70^\circ) \). As a tension test is considered in this work, \( \lambda_1 \) is greater than 1. In the case of \( \beta = 70^\circ \), it is observed that \( J_4 \) is first lower than 1 (the biological tissue is first shortened in the fiber direction) and decreases to a minimum value. This case has been previously discussed in Section 3.2 (Case c2) and can only occur if \( \beta > \beta_c \). The minimum value of \( J_4 \) is equal to 0.85 according to (45). This minimum is reached if \( \lambda_1 \) is approximately equal to 1.56 (Fig. 8 a) and corresponds to a deformed angle \( \theta \) equal to the critical angle \( \beta_c \) (Fig. 8 b). After this minimum value, \( J_4 \) increases and becomes greater than 1 if \( \lambda_1 \) is approximately greater than 2.29. This value indicates the transition from a purely isotropic to an anisotropic behavior of biological tissues. It can be analytically calculated by using \( \lambda_1 = (\sqrt{14+4f^2-1)/2} \). This formula is easily obtained by solving (37) together with \( J_4 = 1 \). It is worth noting that if \( \beta < \beta_k \) \( (e.g., \beta = 20^\circ) \), \( J_4 \) increases with \( \lambda_1 \) (Fig. 8 a) so that the biological tissue is always stretched in the fiber direction \( (J_4 \geq 1) \). The critical angle \( \beta_c \) plays obviously a key role for the understanding of the model behavior. However, it should be noted that the minimum value of \( J_4 \) and the correspondence between the solution and \( J_4 \) were only discussed in the case of \( J_4 < 1 \). As the orthotropic part of the HGO energy density is assumed to generate no stress if \( J_4 < 1 \), the critical angle concerns only the neo-Hookean part of the model. The results related to \( \beta \) given in this paper could thus be extended to any anisotropic...
model superposing any orthotropic density with the isotropic neo-Hookean density.

The HGO model has been implemented into an in-house finite element code FER [14]. Fig. 9 shows a good agreement between the numerical results obtained with FER and the analytical calculation presented in this work with \( \beta = 70^\circ \). Beyond a threshold value of \( J_4 \approx 2.29 \), it is observed that \( \lambda_1 \) increases while \( \lambda_2 \) decreases. In other words, the section shortens in the direction 2 and swells in the direction 3 as shown in Fig. 10. As outlined in [12], this trend typically represents an anisotropic behavior. Before this transition, the HGO model reduces to the isotropic neo-Hookean model and the relationship between \( \lambda_1 \) and \( \lambda_2 \) is defined by the neo-Hookean solution (36).

5. Conclusions

In this paper, a theoretical study has been performed with Holzapfel–Gasser–Ogden’s (HGO) constitutive law and applied to an unconstrained tension test. The free traction loading condition related to this test is expressed as a non-linear algebraic equation to give a relationship between the principal stretches \( \lambda_1 \) and \( \lambda_2 \). To change this equation into a cubic polynomial form, we propose to use a cylindrical coordinate system by setting the square root of the mixed invariant \( J_4 \) as the radial coordinate. The deformed angle \( \theta \), between the collagen fibers and the circumferential direction, is used as the polar angle. A closed form solution has been obtained thanks to Cardano’s formula. A critical angle \( (\beta_c \approx 54.73^\circ) \) has been analytically determined.
References


